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First Semester M.Sc. Degree Examination, May/June - 2023

(NON CBCS) (SLM Scheme)

MATHEMATICS

Algebra

Time : 3 Hours **Max. Marks : 80**

Instructions : 1) Answer any five full questions.

2) All questions carry equal marks.

Q1) a) Define a subgroup of a group. Let H and K be subgroup of a group G. Then prove that HK is a subgroup of G, if and only if, HK = KH.

b) State and prove the fundamental theorem of homomorphism for groups.

c) If G' is a commutator subgroup of a group G then prove that G' is normal in G and G/G' is abelian.

[6 + 5 + 5]

Q2) a) Let H and N be subgroups of G and let N be normal in G. Then prove that HN is a subgroup of G, N is normal in HN and H∩N is normal in H. Further, $\frac{HN}{N} \cong \frac{H}{H \cap N}$.

b) State and prove the Cayley's theorem for permutation.

c) Prove that for $n \geq 2$, the map $\varepsilon : S_n \to \{1, -1\}$ defined by signature is a homomorphism of the group S_n onto the multiplicative group of two elements $\{1, -1\}$.

[5 + 6 + 5]

Q3) a) Derive the class equation.

b) Let G be a finite group and let P^r divide O(G) where P is a prime. Then show that G has a subgroup of order P^r.

c) Show that a group of order $5^2.7^2$ is abelian.

[5 + 7 + 4]

Q4) a) Define an integral domain. Show that every integral domain can be embedded in a field.

b) Let R be a commutative ring with identity. Then prove that an ideal M is a maximal ideal in R if and only if R/M is a field.

[10 + 6]

Q5) a) Let a and b be nonzero elements of R. Then show that *a* and *b* are associates if and only if *a* = ub for some unit *u* in R.

b) Define a Euclidean domain. Show that every Euclidean domain is a principal ideal domain.

c) If p is a prime number such that $p \equiv 1 (\text{mod } 4)$, then prove that p is a sum of two squares of integers.

[4 + 7 + 5]

Q6) a) Prove that in a unique factorization domain R, an element is a prime if and only if it is irreducible.

b) If F is a field, then show that F(X) is a Euclidean ring.

c) State and prove Eisenstein's criterion.

[6 + 5 + 5]

Q7) a) Let S be a linearly independent subset of a vector space V and let V be a vector in V such that $v \notin$ S. Then show that S $\cup\{v\}$ is linearly dependent if and only if $v \in$ L(s).

b) Define a vector space. Show that a vector space generated by a finite set has a finite basis.

c) Show that the characteristic of a field is either 0 or a prime number.

[6 + 7 + 3]

Q8) a) Define normal extension of a field. Prove that any splitting field K of a polynomial over F is a normal extension of F and conversely a finite normal extension of F is a splitting field of some polynomial over F.

b) Show that the polynomial $x^3 - 2$ is separable.

c) Show that any finite separable extension is a simple extension.

[8 + 3 +5]

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