Math-1.3

Sl.No. :

Total No. of Pages : 3

First Semester M.Sc. Degree Examination, May/June - 2023

(NON CBCS) (SLM Scheme)

MATHEMATICS

Complex Analysis - I

Time : 3 Hours **Max. Marks : 80**

Instructions : 1) Answer any five full questions.
2) All questions carry equal marks.

Q1) a) If z_1 and z_2 are complex numbers, then prove that

i) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ and

ii) $\left(\overline{\frac{z_1}{z_2}}\right) = \frac{\overline{z_1}}{\overline{z_2}}, \left(z_2 \neq 0\right)$

b) Let z_1 and z_2 be two vertices of a square. Then find the other vertices in all possible cases.

c) Derive the general equation of a straight line.

[4 + 6 + 6]

Q2) a) If z_1, z_2, z_3 are the vertices of an isosceles right angled triangle with right angle at z_2 then prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2\left(z_1 + z_3\right)$.

b) Evaluate $\lim_{z \to e^{i\pi/3}} \frac{\left(z - e^{i\pi/3}\right)z}{\left(z^3 + 1\right)}$.

c) Find all points of discontinuity for the following functions.

i) $\dfrac{\operatorname{Re} z}{|z|}$ ii) $\sqrt[n]{z}$

iii) $\arg\left(1-\sqrt{z}\right)$ iv) $\sqrt{1-z^2}$

[4 + 4 + 8]

Q3) a) State and prove the sufficient condition for a function $f(z)$ to be analytic.

b) Show that $f(z)=\sqrt{r}\left(\cos\dfrac{\theta}{2}+i\sin\dfrac{\theta}{2}\right)$ where $r > 0$ and $0 < \theta < 2\pi$, is differentiable and find $f^1(z)$.

[8 + 8]

Q4) a) Show that the function $u(x,y)=\dfrac{1}{2}\log\left(x^2+y^2\right)$ is harmonic. Determine its harmonic conjugate and the corresponding analytic function.

b) State and prove Weierstrass M-test for convergence of series of functions.

c) If $\sum_{n=0}^{\infty} a_n$ converges then prove that $f(z)=\sum_{n=0}^{\infty} a_n z^n$ tends to $f(1)$ as $z \to 1$ in such a manner that $\dfrac{|1-z|}{|1-|z||}$ remains bounded.

[4 + 4 + 8]

Q5) a) Find the region of convergence of the following power series.

i) $\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)z^n}{(n+2)(n+3)}$

ii) $\displaystyle\sum_{n=1}^{\infty}\frac{n!z^n}{n^n}$

b) Prove that $e^z = 1$ if and only if $z = 2k\pi i$, $k \in Z$.

c) Define logarithmic function. Prove that $\log z$ can have real value if and only if z is positive.

[4 + 4 + 8]

Q6) a) Prove that bilinear transformation other than identity transformation has atmost two fixed points.

b) Define cross ratio. Prove that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie on a circle or on a straight line.

[8 + 8]

Q7) a) Evaluate $\int_\gamma |z| dz$, where $\gamma(t) = te^{it}$, $0 \le t \le \pi$.

b) If a function f is continuous in an open set U and analytic in $U - \{z_0\}$ for some point z_0 of U then prove that $\int_{\partial R} f(z)dz = 0$ for every closed rectangle R in U.

c) State and prove Cauchy's theorem for disk.

[3 + 5 + 8]

Q8) a) State and prove Cauchy's integral formula and hence prove that the value of an analytic function $f(z)$ at a point on the domain D equals the average of its value on the bounded area of the circle $|z - a| = p$ contained in D.

b) Evaluate $\int_C \frac{dz}{(z-i)(z+i)}$, where $C : |z + i| = 1$.

c) Find and classify the isolated singularities of $\frac{\sin z}{z^2}$.

[10 + 4 + 2]

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