Math-2.1

Sl.No. :

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II Semester M.Sc. Degree Examination, June - 2023
(Non CBCS, SLM Scheme)
MATHEMATICS
Linear Algebra

Time : 3 Hours **Max. Marks : 80**

Instruction: 1) ***Answer any five full questions.***

2) ***All questions carry equal marks.***

Q1) a) Define a vector space with an example. Explain the general properties of vector space.

b) Prove that the set S ={(1, 0, 0, –1), (0, 1, 0 –1), (0, 0, 0, 1)} is linearly independent.

c) Define standard basis and find the dimension of vector spaces F_n, $M_{m\times n}$ (F) and P_n (F). **[6+6+4]**

Q2) a) Define a linear transformation. Show that product of any two linear transformations from a vector space into itself is also linear.

b) State and prove dimension theorem for linear transformation.

c) Define algebra of linear transformation and explain its properties.

[6+6+4]

Q3) a) If V is finite dimensional vector space over a field F, then prove that $T \in A(V)$ is non-invertible (singular) if and only if there exists a $v \neq 0$ in V such that $T(v) = 0$.

b) If T has matrix $A = [a_{ij}]_{m\times n}$ relative to the bases A of U and B of V. If Q is the matrix of transition from A to the basis A′ of U and P is the matrix of transition from B to the basis B′ of V, then prove that the matrix of T relative to A′ and B′ is $P^{-1}AQ$.

c) Prove that the two $m\times n$ matrices A and B represent the same linear transformation T of U into V if and only if A and B are equivalent.

[4+6+6]

Q4) a) If A = $[a_{ij}]_n$, then prove that det(A^T) = det (A).

b) If a matrix B results from a matrix A by interchanging two rows (columns) of A, then prove that det (B) = – det A.

c) Define minors and cofactors. Solve the following system of linear equations by Cramer's rule.

$-2x + 3y - z = 1$

$x + 2y - z = 4$

$-2x - y + z = -3$

[4+5+7]

Q5) a) Show that a square matrix A has 0 (zero) as eigen values if and only if A is not invertible.

b) Let T be a linear operator on a finite dimensional vector space V and let $f(t)$ be the characteristic polynomial of T. Then show that $f(T) = T_0$, the zero transformation.

c) Let V be an inner product space over F. Then prove the following :

i) $\left|\langle x, y\rangle\right| = \|x\| \cdot \|y\|$

ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

[5+5+6]

Q6) a) Let w_1 = (1, 0, 1, 0), w_2 = (1, 1, 1, 1) and w_3 = (0, 1, 2, 1) in $\mathbb{R}^4$. If $\{w_1, w_2, w_3\}$ is linearly independent. Use the Gram-Schmidt orthogonalization process to compute the orthogonal vectors v_1, v_2 and v_3 and then find the normalize of these vectors to obtain an orthonormal set.

b) Let C [–π, π] be the inner product space of all continuous functions defined on [–π, π] with the inner product defined by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t) \cdot g(t)\, dt$. Prove that sin t and cos t are orthogonal functions in C [–π, π].

c) State and prove Bessel's inequality.

[6+4+6]

Q7) a) Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then show that there exists a unique function T* : V $\rightarrow$ V such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in$ V. Furthermore, T* is linear.

b) Let V be an inner product space and let T be a linear operator on V. Then prove that T is an orthogonal projection if and only if T has an adjoint T* and $T^2 = T = T^*$.

c) Determine the rank and signature of the real quadratic form matrix

$$A = \begin{bmatrix} 4 & 5 & -2 \\ 5 & 1 & -2 \\ -2 & -2 & 3 \end{bmatrix}.$$

[6+6+4]

Q8) a) If $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let T_1 and T_2 be the linear transformations induced by T on V_1 and V_2 respectively. If the minimal polynomial for T_1 is $p_1(x)$ and T_2 is $p_2(x)$, then prove that the minimal polynomial for T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

b) Prove that the linear transformation T is diagonalizable if and only if its minimal polynomial is a product of distinct linear factors.

c) Show that the characteristic polynomial of the companion matrix C($q(x)$) is $q(x)$.

[7+4+5]

