Math-2.2

Sl.No. :

Total No. of Pages : 3

II Semester M.Sc. Degree Examination June - 2023

(Non - CBCS, SLM Scheme)

MATHEMATICS

Real Analysis - II

Time : 3 Hours **Max. Marks : 80**

Instructions: ***1) Answer any five full questions.***

2) All quetions carry equal marks.

Q1) a) Let $A \subseteq \mathbb{R}$, f and g be functions from A into $\mathbb{R}$. If $x_0 \in \mathbb{R}$ then prove that $\lim_{x \to x_0}(fg)(x) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right)$

b) State and prove sequential criterion for continuity.

c) Let $A \subset \mathbb{R}$ and $f_1, f_2, ..., f_k : A \to \mathbb{R}$ are k functions show that a function $\vec{f} : A \to \mathbb{R}^k$ defined by $\vec{f}(x) = (f_1(x), f_2(x), ..., f_k(x))$ is continuous if and only if each of the functions $f_1, f_2, ..., f_k$ is continuous.

[6+6+4]

Q2) a) Define a uniform continuous function. Verify uniform continuity for the function $f(x) = \frac{1}{x}$ on A = (0.1).

b) If f is a continuous mapping of a compact space A in $\mathbb{R}$ into $\mathbb{R}^k$, then show that f(A) is closed and bounded and thus f is bounded.

c) Prove that continuous image of a compact space is compact.

[5+6+5]

Q3) a) State and prove Darboux's theorem.

b) State and prove L Hospital's rule.

c) Find $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

[5+8+3]

Q4) a) Prove that the function f is Riemann - Stieltje's integrable on $[a, b]$ if and only if for every $\in > 0$ there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \in$.

b) Show that $f(x) = x^2 \in \mathbb{R}[x^2]$ on $[0,1]$

c) Show that $f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$ is not Riemann - Stieltjes integrable on $[a, b]$ for any monotonically increasing function $\alpha(x)$ on $[a, b]$.

[6+6+4]

Q5) a) If $f \in \mathbb{R}(\alpha)$ on $[a, b]$ then show that $-f \in \mathbb{R}(\alpha)$ on $[a, b]$ and $\int_a^b (-f)\, d\alpha = -\int_a^b f d\alpha.$

b) Let f be a bounded real function on $[a, b]$ and let α be monotonically increasing on $[a, b]$ such that α' is Riemann integrable on $[a, b]$. Then show that $f \in \mathbb{R}(\alpha)$ on $[a, b]$ if and only if $f\alpha'$ is Riemann integrable function on $[a, b]$ and $\int_a^b f dx = \int_a^b f(x)\alpha'(x)\, dx.$

[8+8]

Q6) a) Let $f \in \mathbb{R}(\alpha)$ on [a , b] and let $g(x) = \int_a^x f(t)dt \quad \forall x \in [a\ ,\ b]$. Then prove that $g(x)$ is continuous on [a , b]. Further show that if f is continuous at $x_0 \in [a\ ,\ b]$ then g is differentiable at x_0 and $g'(x_0) = f(x_0)$.

b) State and prove second mean value theorem for Riemann - Stieltjes Integrals.

c) State and prove fundamental theorem of calculus.

[8+4+4]

Q7) a) Show that a function of bounded variation f on [a , b] can be expressed as a difference of two monotonically increasing functions on [a , b] and conversely.

b) State and prove Cauchy criterion for uniform convergence of a sequence of real valued functions.

c) Prove that the series $\sum_{n=1}^{\infty} \frac{(\cos nx)}{n^2}$ converges uniformly on R.

[7+7+2]

Q8) a) State and prove stone Weierstrass theorem.

b) Define power series. Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

[10+6]