Math - 3.3

Sl.No. :

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III Semester M.Sc. Examination, May - 2023
(Non-CBCS SLM Scheme)
MATHEMATICS
Functional Analysis

Time : 3 Hours **Max. Marks : 80**

Instructions : ***1) Answer any five questions.***
2) All questions carry equal marks.

Q1) a) Define a closed set in a metric space. Show that a set F is a closed set in a metric space if and only if the complement of F is open.

b) If $\overline{A}$ denotes the closure of a subset A of a metric space (X,*d*) then show that $\overline{A} = \{x \in X : d(x, A) = 0\}$.

c) Let (X,d) be a complete metric space and (Y,*d*) be a subspace of (X,*d*). Prove that Y is closed if and only if Y is complete.

[6+4+6]

Q2) a) If (*x*,*d*) is a complete metric space and if T : X$\rightarrow$X is a coutraction mapping, show that T has a unique fixed point.

b) State and prove cautor's intersection theorem

[8+8]

Q3) a) Define Bolzano-Weierstrass property. if (X,*d*) is a metric space, show that X is a sequentially compact if and only if X has Bolzano-Weierstrass property.

b) Show that $\mathscr{B}$(N,N') is a linear space. Define ||T||=sup {||T*x*|| : ||*x*||$\leq$|}. Under this norm show that $\mathscr{B}$(N,N') is also a normed linear space.

[10+6]

Q4) a) State Weierstrass theorem and Stone-Weierstrass theorem in real case. Deduce Weierstrass theorem as a simple corollary to Stone-Weierstrass theorem.

b) State and prove Hahn-Banach theorem for complex linear space.

[8+8]

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Q5) a) Let N be a normal linear space and $0 \neq x_0 \in N$. Show that there exists a functional f_0 in N* such that $f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$.

b) Let X be a normad linear space. if the closed unit ball $B = \{x \in X : \| x \| \leq 1\}$ in X is compact, then prove that X is finete dimensional.

[8+8]

Q6) a) State and prove open mapping theorem.

b) If T : X$\rightarrow$Y is one-to-one continuous linear operator from a Barach space X onto Barach space Y, show that $T^{-1} : Y \rightarrow X$ is also a continuous linear operator.

[10+6]

Q7) a) State and prove closed graph theorem.

b) If N is a normed linear space, show that a non-empty subset X of N is bounded if and only if $f(x)$ is a bounded set of numbers for every $f \in N^*$.

[10+6]

Q8) a) Let M be a closed linear subspace of a Hilbert space H. Prove that $H = M \oplus M^{\perp}$.

b) If H is a Hilbert space and $S \subseteq H$, show that $S^{\perp}$ satisfies the following properties.

i) $S \cap S^{\perp} = \{0\}$

ii) $S^{\perp}$ is a closed subspace of H

iii) $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

iv) $S \subseteq S^{\perp\perp}$ Where $S^{\perp}$ is the orthogonal complement of S.

[8+8]

