MATH3.3

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III Semester M.Sc. Degree Examination, May - 2026
(Scheme : Non - CBCS)
MATHEMATICS
Functional Analysis

Time : 3 Hours **Max. Marks : 80**

Instructions: 1) Answer any five full questions.
2) All questions carry equal marks.

Q1) a) Define metric space. Prove that a set G is open in a metric space (X,d) if and only if G is a union of open spheres.

b) Prove that every closed subspace of a complete metric space is also complete.

[10+6]

Q2) a) Define a contraction mapping Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping. Then prove that T has a fixed point.

b) State Cantor's intersection theorem and Baire's category theorem. Making use of Cantor's intersection theorem prove Baire's category theorem.

[8+8]

Q3) a) If (X,d) is a metric space then show that X is sequentially compact if and only if X has Balzano-Weierstrass property.

b) If (X,d) is a metric space, then prove that the following are equivalent.

i) X is compact

ii) X is sequentially compact

iii) X is complete and totally bounded

[8+8]

Q4) a) Let a linear space X be the sum of two subspaces M and N, so that X=M+N. Then prove that $X=M \oplus N \Leftrightarrow M \cap N=\{0\}$.

b) Let X be a normed linear space, then prove that the closed unit ball $B=\{x \in X: \|x\| \leq 1\}$ in X is compact if and only if X is finite dimensional.

[6+10]

Q5) a) State and prove Hahn - Banach theorem for complex linear space.

b) If X is a compact Hansdorff space, then prove that $C(X, \mathbb{R})$ separates points of X.

[12+4]

Q6) a) State and prove open mapping theorem.

b) IF $T: X \to Y$ is one to one continuous linear operator from a Banach space X onto Banach space Y, then show that $T^{-1}: Y \to X$ is also a continuous linear operator.

[12+4]

Q7) a) If N is a normed linear space, show that a non-empty subset X of N is bounded if and only if $f(X)$ is a bounded set of numbers for every $f \in N^*$.

b) Prove the following two properties of Hilbert space H:

i) $\|X+Y\|^2+\|X-Y\|^2=2\left(\|X\|^2+\|Y\|^2\right)$

ii) If $X_n \to X, Y_n \to Y$ as $n \to \infty$, then $(X_n, Y_n) \to (X, Y)$ as $n \to \infty$.

c) If H is a Hilbert Space and $S \subseteq H$. Then show that $S^{\perp}$ satisfies the following properties.

i) $S \cap S^{\perp}=\{0\}$

ii) $S^{\perp}$ is a closed subspace of H

[6+5+5]

Q8) a) Prove that the set of all self adjoint operators is a closed (real) linear subspace of B(H) and therefore is a real Banach space.

b) Show that an operator T on H is self adjoint if and only if $(T(x), x)$ is real for all x.

[8+8]

