MMDSC-1.1

Sl. No.

I Semester M.Sc. (CBCS) Degree Examination, October/November - 2025
MATHEMATICS
Algebra - I
Course : I

Time : 3 Hours **Maximum Marks : 80**

Instructions to Candidates:

Answer any Five full questions. All questions carry equal marks.

1. a) Define : i) Normal subgroup, and ii) quotient group of a group. Let N be a normal subgroup of a group G. Then prove that

i) $N \Delta G \Leftrightarrow gNg^{-1} = N \ \ for\ all\ g \in G$

ii) $N \Delta G \Leftrightarrow Ng = gN \ \ for\ all\ g \in G$

iii) $N \Delta G \Leftrightarrow Na\ Nb = Nab \ \ for\ all\ a, b \in G$

b) State and prove the fundamental theorem of homomorphism for groups. **(10+6)**

2. a) Let H and K be normal subgroups of a group G such that $K \subseteq H$ .Then show that H/K is normal in G/K and $(G/K)(H/K) \cong G/H$.

b) State and prove Cayley's theorem for permutation groups.

c) Prove that any non-abelian group of order 6 is isomorphic to symmetric group S_3. **(6+6+4)**

3. a) If H is a subgroup of G. Let $N(H) = \left\{ g \in G / gHg^{-1} = H \right\}$. Show that,

i) $N(H)$ is a subgroup G

ii) H is a normal in $N(H)$

iii) If H is a normal subgroup of the subgroup K in G, then $K \subset N(H)$.

b) Let G be a finite group. Then prove that $o(G) = o\left(Z(G)\right) + \Sigma_{[G,N(a)]>1}\left[G : N[a]\right]$

c) Show that for any group of order p^n , where p is prime, has nontrivial center. **(6+5+5)**

[P.T.O.

4. a) Define *p*-sylow subgroups of a finite group.

If *p* is a prime number and p^{α} |o(G), $p^{\alpha+1}$ +o(G), then show that G has a subgroup of order p^{α}.

b) Find all the Sylow *p* - subgroups of the symmetric group S_4 and show none of them is normal. **(12+4)**

5. a) Define the following with an example:

i) Ring;

ii) Quotient Ring;

iii) Integral Domain;

iv) Division Ring; and

v) Field.

b) Let R and R^1 be rings and ϕ is homomorphism of R onto R^1 with Kernel *U*. Then prove that $R^1 \approx R/U$. Further, show that there is one to one correspondence between the set of ideals W^1 of R^1 and the set of ideals *W* of *R* containing *U*.

c) Show that the homomorphism $\phi : R \to R^l$ is an isomorphic if and only if $K\text{er}\,\phi = \{0\}$. **(5+7+4)**

6. a) Define the principal ideal and principal ideal ring. Show that a ring *Z* of integers is a principal ideal ring.

b) If *R* is a commutative ring with unity and *M* is an ideal of *R*, then show that *M* is a maximal ideal of *R* if and only if *R/M* is a field.

c) Show that for every integral domain can be imbedded into a field. **(4+5+7)**

7. a) Define an Euclidean ring. Prove that the ring Z[i] of Gaussian integers is an Euclidean ring.

b) Show that $Z=[\sqrt{2}]=\{a+\sqrt{2}b : a,b \in z\}$ is an euclidean domain.

c) If *p* is a prime number of the form *4n+1*, then prove that $p = a^2 + b^2$ for some integers *a* and *b*. **(6+4+6)**

8. a) If the ring *R* is an integral domain, then show that *R[x]* is also an integral domain.

b) If *F* is a field, then show that *F[x]* is not a field.

c) State and prove the Eisenstein criterion for the irreducibility of a polynomial. **(5+5+6)**

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