MMDSC-1.2

Sl.No. :

Total No. of Pages : 3

I Semester M.Sc. Degree Examination, May/June - 2023

(CBCS, SLM Scheme)

MATHEMATICS

Real Analysis - I

Time : 3 Hours **Max. Marks : 80**

Instructions: ***1) Answer any five full questions.***
2) All questions carry equal marks.

Q1) a) State and prove Archimedean property.

b) Prove that for every real $x > 0$ and every integer $n > 0$, there is one and only one real y such that $y^n = x$.

c) Define a countable set. Prove that every subset of a countable set is countable.

[3+6+7]

Q2) a) Prove that the set E in a metric space X is open if and only if its complement is closed.

b) Let X be a metric space and $E \subset X$.

Then prove that,

i) $\overline{E}$ is closed

ii) $E = \overline{E}$ iff E is closed.

iii) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

c) Let X be a metric Space. Suppose $K \subseteq Y \subseteq X$ then K is compact relative to X if and only if K is compact relative to Y.

[5+6+5]

Q3) a) Prove that in a metric space every K-cell is compact.

b) Prove that the sub-sequential limits of a sequence $\{x_n\}$ form a closed subset of $\mathbb{R}$.

c) Let X be a metric space and $E \subset X$. If x is a limit point of E, then prove that there is a sequence $\{x_n\}$ in E that converges to x.

[6+6+4]

Q4) a) Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Also show that e is irrational.

b) Prove that the series $\sum_{n=0}^{\infty}\frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \le 1$.

[8+8]

Q5) a) Let $A, B \subset \mathbb{R}$, $f: A \to \mathbb{R}$, $g: f(A) \to \mathbb{R}$ and $h: A \to \mathbb{R}$ defined by $h(x) = g(f(x)), \forall x \in A$. Show that if f is continuous at the point $x_0 \in A$ and g is continuous at $f(x_0)$ then h is continuous at x_0.

b) If f is a continous mapping of a compact space A in $\mathbb{R}$ into $\mathbb{R}^k$. Then show that $f(A)$ is closed and bounded and then f is bounded.

c) Show that $\lim_{x\to a}\frac{1}{x} = \frac{1}{a}$ if $a > 0$.

[6+6+4]

Q6) a) State and prove Intermediate value theorem.

b) Let f be menotonically increasing on (a, b) then $f(x+)$ and $f(x-)$ exist at every point of (a, b) more precisely.

$\sup f(t) = f(x-) \le f(x) \le f(x+) = \inf f(t)$

$a < t < x$ $\qquad\qquad$ $x < t < b$

Furthermore, if $a < x < y < b$ then prove that $f(x+) \le f(y-)$

c) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous map. If E is a connected subset of $\mathbb{R}$ then $f(E)$ is also connected.

[5+6+5]

Q7) a) State and prove L' Hospital's Rule.

b) If f and g are real valued continous function defined on $[a, b]$ and differentiable in (a, b) then prove that there is a point $x \in (a, b)$ such that.

$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x)$.

c) Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

[7+7+2]

Q8) a) State and prove Darboux's theorem.

b) If $0 < x < 2\pi$, prove that

i) $\text{Sin } x < x$

ii) $\text{Cos } x > 1 - \frac{x^2}{2!}$

c) State and prove Taylor's theorem.

[4+4+8]