MMDSC-1.3

Sl.No. :

Total No. of Pages : 3

I Semester M.Sc. Degree Examination, May/June - 2023

(CBCS, SLM Scheme)

MATHEMATICS

Complex Analysis - I

Time : 3 Hours **Max. Marks : 80**

Instructions: ***1) Answer any five full questions.***
2) All questions carry equal marks.

Q1) a) For $z \neq 0$ and $w \neq 0$, demonstrate that $|z + w| = |z| + |w|$ is true if and only if $w = tz$ for some $t > 0$.

b) Prove the following:

i) $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

ii) $\left|\dfrac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right| < 1$ if $|z_1| < 1$ and $|z_2| < 1$.

[8+8]

Q2) a) Show that intersection of arbitrary collection of open sets need not be open.

b) Prove that a set $A \subset \mathbb{C}$ is closed if and only if $A = \overline{A}$. That is if and only if A contains all its boundary points.

c) Show that the boundary ∂U and closed $\overline{U}$ of any subset of $\mathbb{C}$ are always closed sets.

[4+6+6]

Q3) a) Prove that every convergent sequence has a unique accumulation point.

b) Prove that a point z_0 belongs to the closure of a set A if and only if there exists a sequence $\{z_n\}_{n=1}^{\infty}$ converging to z_0

c) If D is open in $\mathbb{C}$, prove that the components of D are always open.

[5+5+6]

Q4) a) Define bounded sets. Show that a subset D of $\mathbb{C}$ is compact if and only if it is closed and bounded.

b) If $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$ are two functions, suppose $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w'$. Then prove the following:

i) $C.\lim_{z \to z_0} f(z) = \lim_{z \to z_0} C.f(z)$

ii) $\mathrm{Re}(z) \to \mathrm{Re}(w_0)$ and $\mathrm{Im}(z) \to \mathrm{Im}(z_0)$ as $z \to z_0$

iii) $\overline{f(z)} \to \overline{w}_0$ as $z \to z_0$

iv) $|f(z)| \to |w_0|$ as $z \to z_0$

v) $f(z) + g(z) \to w_0 + w'$.

[6+10]

Q5) a) Evaluate, $\lim_{z \to e^{\frac{i\pi}{3}}} \left(z - e^{\frac{i\pi}{3}} \right) \frac{z}{\left(z^3 + 1 \right)}$.

b) State and prove the sufficient conditions for $f(z)$ to be analytic.

[8+8]

Q6) a) State and prove Cauchy - Hadamard Theorem.

b) Prove that the power series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ obtained by differentiating the power series $\sum a_n z^n$ has same radius of convergence as the original series $\sum a_n z^n$.

[10+6]

Q7) a) Evaluate $\int_C \frac{dz}{z-a}$, where C is any simple closed curve. Considering two cases when

i) z = a is outside C.

ii) z = a is inside C.

b) State and prove Cauchy's Integral formula.

c) Evaluate $\int_C \frac{z^3-2z+1}{(z-i)^2}dz$, where C is the circle $|z|=2$.

[5+7+4]

Q8) a) If each function in a sequence $\{f_n\}$ is continuous on a set A and that this sequence converges uniformly on A to the limit function f. Then prove that f is also continuous on A. Furthermore show that, $\int_\gamma f(z)dz = \lim_{n\to\infty}\int_\gamma f_n(z)dz$ for every piecewise smooth path γ in A.

b) Determine the Laurent series representation of $f(z)=(z-1)^{-3}\sin z$ in the ring $D=\{ z : 0 < |z-1| < \infty\}$

[10+6]