MMDSC-2.1

Sl. No.

II Semester M.Sc. Degree Examination, December - 2023
MATHEMATICS
Algebra-II
(CBCS Scheme)

Time : 3 Hours **Maximum Marks : 80**

***Instructions to Candidates*:**

Answer any Five full questions. All questions carry equal marks.

1. a) Let A be a non-zero ring. Then, show that the following are equivalent.

i) A is a field

ii) The only ideals in A are $<0>$ and $<1>$.

iii) Every homomorphism of A into a non-zero B is injective.

b) Define a local ring with an example. Prove that $\mathbb{Z}_n$ is a local ring if and only if $n>1$ is a power of a prime. **(6+5+5)**

c) Prove that the sum and product of finitely generated ideal is again finitely generated.

2. a) Write a short note on Prime Spectrum of a ring.

b) Let M be an A-Module. Then show that $Hom_A(A,M) \cong M$.

c) Let N_1, N_2 be submodules of an A-module M. Prove that the following.

i) $Ann(N_1 + N_2) = Ann(N_1) \cap Ann(N_2)$.

ii) $(N_1 : N_2) = Ann((N_1 + N_2)/N_1)$. **(3+6+7)**

3. a) Let M be a finitely generated A-Module, Let I be an ideal of A, and Let ϕ be an A-module endomorphism of M such that $\phi(M) \subseteq IM$. Then, show that ϕ satisfies an equation of the form $\phi^n + a_1\phi^{n-1} + ... + a_{n-1}\phi + a_n = 0$, where $a_i \in I, 1 \le i \le n$.

[P.T.O.

b) Show that every abelian group is a Z-module.

c) Define the following with example.

i) Exact sequence.

ii) A commutative diagram. **(8+4+4)**

4. a) Let M be an *A-module*. Show that the following are equivalent.

i) M is a Noetherian A-module.

ii) Every nonempty class of submodules of M has a maximal element.

iii) Every submodule of M is finitely generated.

b) Let A_1 be a Neotherian ring and $f : A_1 \to A_2$ be a surjective ring homomorphism. Prove that every ideal A_2 is finitely generated.

c) If A is a noetherian ring, then show that the polynomial ring A[x] is also a Netherian ring. **(6+4+6)**

5. a) Let K be a finite extension of field F and L be finite extension of K. Prove that L is a finite extension of F and *[L:F] = [L:K] [K:L]*.

b) Prove that the polynomial $f(x) = x^4 - 10x^2 + 1$ is irreducible over $\mathbb{Q}$.

c) Find the degree of extension of the following:

i) $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ over $\mathbb{Q}$

ii) $\mathbb{Q}[\sqrt{2}, i]$ over $\mathbb{Q}$ **(6+4+6)**

6. a) Every non-constant polynomial with complete coefficients has a complex root.

b) Define a splitting field and degree of extension. Find the splitting field and degree of extension of $f(x) = x^2 + 1$ over $\mathbb{Q}$. **(10+6)**

7. a) Show that the set of all constructible numbers form a subfield of $\mathbb{R}$.

b) If K be a finite field of characteristic p. Prove that K has a subfield isomorphic to F_p.

(8+8)

8. a) Let $ChF = p > 0$ and $f(x) \in F[x]$ a monic irreducible polynomial. Prove that, $f(x)$ is in separable if and only if $f(x)$ is a polynomial in x^p.

b) Let K/L be a finite separable extension. Show that, K=F (α) for some $\alpha \in K$.

c) Show that, if K be an algebraic extension of a field F that is not a finite extension of F, then there exist elements in K where degrees over F are arbitrarily large. (6+6+4)
