MMDSC - 2.1

Sl.No. : **Total No. of Pages : 2**

II Semester M.Sc. Degree Examination, May - 2024
(CBCS, SLM Scheme)
MATHEMATICS
Algebra-II

Time : 3 Hours **Max. Marks : 80**

Instructions: ***1) Answer any five full questions.***
2) All questions carry equal marks.

Q1) a) Define a local ring. If A is a ring & M is a maximal ideal of A such that every element 1+M is a unit in A then show that A is a local ring and M is the unique maximal ideal of A.

b) Define extension and contraction of ideals with respect to a ring homomorphism $f : A \to B$. If C denote the set of all contracted ideals I of A & E denote the set of all extended ideals J of B, show that $C = \{I / I^{ec} = I\}$ & $E = \{J / J^{ce} = J\}$. Also prove that $I \to I^{e}$ is a bijective map C into E when inverse is $J \to J^{c}$.

c) Suppose I is an ideal with $I \subseteq \bigcup_{i=1}^{n} P_i$, where P_i are prime ideals. Then show that $I \subseteq P_i$ for some *i*.

[4+7+5]

Q2) a) Define module & module homomorphism. Let M be an A-module then prove that $\mathrm{Hom}_A(A,M) \cong M$.

b) For a prime ideal P of ring A, describe the localization A_P of A with respect to P, show that A_P is a local ring.

[8+8]

Q3) Prove that the sequence $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ is exact if and only if $0 \to \mathrm{Hom}(M'',N) \xrightarrow{\bar{v}} \mathrm{Hom}(M,N) \xrightarrow{\bar{u}} \mathrm{Hom}(M',N)$ is exact for all A-modules N.

[16]

Q4) a) State and prove first isomorphism theorem for modules.

b) Show that a ring A is Noetherian if and only if every prime ideal of A is finitely generated.

[8+8]

Q5) a) Let K/F be field extension & $\alpha \in K$ be algebraic over F. Then prove that there exist unique monic polynomial of smallest degree in F[x] such that $p(\alpha) = 0$. Further if $f(x)$ is any polynomial with coefficients in F such that $f(\alpha) = 0$ then prove that $p(x) / f(x)$ in F[x].

b) i) Find [Q(α,β),Q], where $\alpha = \sqrt[3]{2}$ and β is root of $x^4 + x + 1$.

ii) Find $[Q(\sqrt{2}, \sqrt[3]{2}), Q]$.

c) Find the splitting field and degree of extension of $f(x) = x^4 + 1$ over Q.

[6+5+5]

Q6) a) Let F be a field, $f(x)$ be a non-constant polynomial with co-efficients in F. Then prove that there exist an extension K of F such that $f(x)$ has a root in K.

b) Prove that every non constant polynomial with complex co-efficients has a complex root.

[6+10]

Q7) a) If K is any finite field, then prove that characteristic of K is prime.

b) Prove that the multiplicative group of non zero elements of a finite field is cyclic.

c) Let F be a field of order p^n where p is prime, n is a integer. Then prove that there exist a subfield K of F of order p^d if and only if d / n.

[6+4+6]

Q8) a) Show that $Q(i, \sqrt[3]{2}) = Q(i\sqrt[3]{2})$.

b) Let F be a field of characteristic zero. Then prove that every finite extension of F is a simple extension.

[4+12]