MMDSC-2.1

Sl. No.

II Semester M.Sc. Degree Examination, May/June - 2025
MATHEMATICS
Algebra - II
(CBCS)

Time : 3 Hours **Maximum Marks : 80**

Instructions to Candidates:

Answer any **Five** full questions. All questions carry **equal** marks.

1. a) Define a Nilradical *N(A)* of a ring *A*.
Show that the nilradical of *A* is an ideal of *A*, and the ring *A/N(A)* has no nilpotent element other than 0.

b) Define a Jacobson Radical *J(A)* of a ring *A*.
Let *A* be non-zero ring. Then show that $x \in J(A)$ if and only if $1 - xy$ is a unit in *A* for all $y \in A$.

c) If $I_1, I_2, \ldots\ldots, I_n$ are mutually coprime ideals of a ring $A\left(i.e..I_J + I_K = A \text{ if } j \neq k, 1 \leq j,k \leq n\right)$ then show that $\bigcap_{j=1}^{n} I_j = \prod_{j=1}^{n} I_j$. **(5+6+5)**

2. a) Let *A* be a ring. Show that the following statements are equivalent:

i) X Spec(A) is disconnected.

ii) A contains an idempotent $\neq 0, 1$.

b) State and prove the first isomorphism theorem for modules.

c) Define an Annihilator of a submodule.
Let N_1, N_2 be submodules of an *A*-module *M*.Prove that $(N_1 : N_2) = Ann\,((N_1 + N_2)/N_1)$.

(6+5+5)

3. a) Show that an *A*-module *M* is finitely generated if and only if *M* is isomorphic to a quotient of A^n for some $n \in N$.

b) State and Prove the Nakayama Lemma.

c) If the sequence $M' \xrightarrow{u} M \xrightarrow{v} M'' \xrightarrow{\circ} 0$ is exact, then prove that for any A-module N, the sequence $O \xrightarrow{\circ}$ Hom $\left(M'', N\right) \xrightarrow{\bar{v}} Hom\left(M, N\right) \xrightarrow{\bar{u}} Hom\left(M', N\right)$ is exact. **(5+6+5)**

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4. a) Let M be an A-module. Then prove following are equivalent:

i. M is a Noetherian A-module.

ii. Every non empty class of submodules of M has a maximal element.

iii. Every submodule of M is finitely generated.

b) If A is a Noetherian ring, then show that the polynomial ring $A[x]$ is also a Noetherian ring. **(8+8)**

5. a) Define the degree of an extension K of a field F.

Let L be an finite extension of field K and K be a finite extension of F. Then show that L is a finite extension of F. Moreover $[L : F] = [L:K]\,[K: F]$.

b) Let K/F be field extension $\alpha, \beta \in K$ be algebraic over K. If $\deg_F(\alpha) = m$ and $\deg_F(\beta) = n$, then prove that $[F[\alpha, \beta]:F] \leq mn$.

c) Show that every finite extension is an algebraic extension. **(6+5+5)**

6. a) Define the splitting field over field F.

Find the splitting field and degree of extension of $f(x) = x^4 + 1$ over Q.

b) Let $f(x)$ be a non-constant polynomial with coefficient in F. Then show that there exists splitting field of $f(x)$ over F.

c) Let K be an algebraic extension of a field F. Then show that K is algebraic closure of F if and only if every non-constant polynomial in $F[x]$ splits in to linear factors over K.

(6+5+5)

7. a) Show that the set of all constructible numbers form a subfield of R**.**

b) Let F be a finite field of characteristic P. Then show that $[F] = p^n$ for some positive integer n.

c) Show that the multiplicative group of non-zero elements of a finite filed is cyclic. **(6+6+4)**

8. a) Let K/F be a finite separable extension. Then show that $K = F(\alpha)$ for some $\alpha \in K$.

b) State and Prove the Galois theorem. **(4+12)**

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