MMDSC-2.1

Sl.No. :

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Second Semester M.Sc. Degree Examination, November - 2024
(CBCS) (SLM Scheme)
MATHEMATICS
Algebra - II

Time : 3 Hours **Max. Marks : 80**

Instruction : ***Answer any Five full questions. All questions carry equal marks.***

Q1) a) Define a Nilradical N(A) of a ring A. Let A be a non-zero ring. Then show that the nilradical of A is the intersection of all prime ideals ofA.

b) Define a contracted and an extended ideal of a ring A.

Let $f : A \to B$ be a ring homomorphism. Let I and J be ideals of A and B. respectively.

Then show that

i) $I \subseteq I^{ec}$, $J \supseteq J^{ce}$

ii) $I^{e} = I^{ece}$, $J^{c} = J^{cec}$

iii) If C denotes the set of all contracted ideals of A and E denotes the set of all extended ideals of B , then $C=\{ I : I^{ec} = I \}$ and $E= \{J : J^{ce} = J\}$.

Further, show that the map $I \to I^{e}$ is bijective map of C onto E, whose Inverse is $J \to J^{e}$.

[6+10]

Q2) a) Write a short note on the following :

i) Prime spectrum of a ring.

ii) Zariski topology.

b) State and prove the second isomorphism theorem for modules.

c) If an *A*-module M is the internal direct sum of its submodules $N_1, N_2, ..., N_k$, then show that M is isomorphic to the external direct sum of A-modules $N_1, N_2, ..., N_k$.

[6+5+5]

Q3) a) Let $f : M \to N$ be a surjectivc A-linear map. If M is finitely generated, prove that N is also finitely generated.

b) Let M be a finitely generated A-module and I be an ideal of A contained in the Jacobson radical $J(A)$ of a ring A. Then $IM = M$ implies $M = 0$.

c) If the sequence $M' \xrightarrow{u} M \xrightarrow{v} M'' \xrightarrow{O} O$ is exact, then prove that for any A module N, the sequence $O \xrightarrow{O} Hom(M'', N) \xrightarrow{\bar{v}} Hom(M, N) \xrightarrow{\bar{u}} Hom(M', N)$ is exact.

[4+4+8]

Q4) a) Define a Noetherian A-module

Let M be a finitely generated module over a Noetherian ring A. Then show that M is a Noetherian A- module.

b) State and prove the Hilbert basis theorem.

[6+10]

Q5) a) Let K/F be field extension and $\alpha \in$ K be algebraic over a field F. Then show that $F[\alpha]$ is a field and $[F[\alpha] : F] = deg\ (F(\alpha))$.

b) Show that every finite extension is an algebraic extension. Justify the converse with suitable example.

c) Find the degree of extension of $\boldsymbol{Q}(\sqrt{2}, i)$ over $\boldsymbol{Q}$.

[5+7+4]

Q6) a) Find the splitting field of the polynomial $x^4 + x^2 + 1$ over **Q** and also degree of extension of the splitting fields over **Q**.

b) State and prove the Kronecker Lemma.

c) Show that no finite field is algebraically closed.

[5+7+4]

Q7) a) Show that a point P(a, b) is constructible point if and only if a and b are constructible real numbers.

b) If K is any finite field, then show that characteristic of K is prime.

c) If m is the exponent of a finite abelian group G, then show that an order of any element of G divides m.

[5+6+5]

Q8) a) Let K/F be a finite separable extension. Then show that $K = F(\alpha)$ for some $\alpha \in K$.

b) State and prove the Primitive element theorem.

c) Show that every quadratic extension is a Galois extension.

[5+7+4]

