MMDSC-2.2

Sl.No. :

Total No. of Pages : 3

II Semester M.Sc. Degree Examination, May - 2024

(CBCS, SLM Scheme)

MATHEMATICS

Real Analysis - II

Time : 3 Hours **Max. Marks : 80**

Instructions: ***1)*** **Answer any five full questions.**

2) All questions carry equal marks.

Q1) a) If P* is a refinement P, then prove that
$L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and $U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

b) If $f_1 \in \mathbb{R}(\alpha)$ and $f_2 \in \mathbb{R}(\alpha)$ on $[a, b]$, then prove that $f_1 + f_2 \in \mathbb{R}(\alpha)$ on $[a, b]$ and $\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$.

c) If $f \in \mathbb{R}(\alpha)$ on $[a, b]$ and if $|f(x)| < \varepsilon$ for all $x \in [a, b]$, then prove that $\left|\int_a^b f d\alpha\right| \leq M(\alpha(b) - \alpha(a))$.

[6+6+4]

Q2) a) State and prove first mean value theorem for Riemann-Stieltjes Integrals.

b) Evaluate $\int_0^3 x\, d([x] - x)$.

[12+4]

Q3) a) State and prove Weierstrass M-test for uniform convergence of series.

b) Let $\{f_n\}$ be a sequence of differentiable functions defined on $[a, b]$ and such that $\{f_n(x)\}$ converges for some point $x \in [a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and $f'(x) = \underset{n\to\infty}{Lt}\ f_n'(x)$ for all $x \in [a, b]$.

c) Show that $\sum_{n=1}^{\infty} \frac{nx}{1+n^2x^2}$ does not converges uniformly on $[a, b]$.

[5+6+5]

Q4) a) State and prove Stone-Weierstrass approximation theorem.

b) Show that the exponential function $E(q) = e^q$, for every rational number q.

[12+4]

Q5) a) Let f be a real bounded function defined on $[a, b]$ and Let α be monotonically increasing on $[a, b]$ then $\int_{-a}^{b} f\, d\alpha \le \int_{a}^{-b} f\, d\alpha$.

b) A function f is Riemann-Stieltje's integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P, such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

c) If f '$[a, b] \to \mathbb{R}$, $f(x) = k$ and $\alpha : [a, b] \to \mathbb{R}$ be monotonically increasing function. Then prove that $f\ \mathbb{R}(\alpha)$ on $[a, b]$.

[6+6+4]

Q6) a) If f is continuous on $[a, +\infty)$ and $\int_a^\infty f(x)dx$ converges absolutely then prove that $\int_a^\infty f(x)dx$ is convergent.

b) Show that with the example if g is continuous and decreasing for $a \le x < \infty$ and $\underset{x\to\infty}{Lt}\, g(x) = 0$, then $\int_a^\infty g(x)\sin x\ dx$ converges.

c) Evaluate $\int_0^\infty e^{-xt}dt$, $1 \le x \le 2$.

[5+6+5]

Q7) a) Prove that $\beta(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, x > 0$ and $y > 0$.

b) Let f be integrable and g be non-negative, decreasing and continuous on $[a, b]$, then there exist $c \in [a, b]$ such that $\int_a^b fg = g(c)\int_a^b f$.

c) Show that $\underset{(x,y)\to(0,0)}{\text{Lim}} \dfrac{xy^3}{x^2+y^6}$ does not exist.

[6+6+4]

Q8) a) State and prove Euler's theorem for homogeneous function.

b) If X is a complete metric space and ϕ is a continuous of X into X. Then there exists one and only $x \in$ X, such that $\phi(x) = x$.

c) State and prove implicit function theorem.

[5+5+6]