MMDSC-3.1

Sl. No.

III Semester M.Sc. Degree Examination, October/November - 2025

MATHEMATICS

Linear Algebra - 1

(CBCS SLM Scheme)

Time : 3 Hours **Maximum Marks : 80**

Instructions to Candidates:

1. ***Answer any Five full questions.***
2. ***All questions carry equal marks.***

1. a) Define a subspace. Let V be the vector space of all real valued continuous functions over the field $\mathbb{R}$ of all real numbers. Show that the set S of solutions of the differential equation $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$ is a subspace of V. **(6+6+4)**

b) Show that the span of any subset S of a vector space V is a subspace of V. Further, show that any subspace of V that contains S must also contain the span of S.

c) Define standard basis and find the dimension of vector spaces F_n, $M_{m\times n}(F)$ and $P_n(F)$.

2. a) Define a linear transformation. Show that product of any two linear transformation from a vector space into itself is also linear. **(6+6+4)**

b) State and prove dimension theorem for linear transformation.

c) Define algebra of linear transformation and explain its properties.

3. a) If V is finite dimensional vector space over a field F, then show that $T \in A(V)$ is non-invertible (singular) if and only if there exists a $v \neq 0$ *in* V such that $T(v) = 0$ **(5+5+6)**

b) If A is an algebra, with unit element, over F, then show that A is isomorphic to a sub algebra of A(v) for some vector space V over F.

c) Define a homogeneous system of equations. If K be the solution set of a system of equations $Ax = b$, Let K_H be the solution set of the corresponding homogeneous system of equation $Ax = 0$ then prove that for any solution S to $Ax = b$, $K = \{s\} + k_H = \{s + k\ ;\ k \in K_H\}$

[P.T.O.

4. a) Define a symmetric matrix. Prove that a matrix A is symmetric if and only if $A^T=A$. **(5+6+5)**

b) Define minors and Co-factors. Explain the Cramer's rule with an example.

c) Define an eigen value of a matrix. If λ is an eigenvalue of an invertible matrix A over a field F, then show that λ^{-1} is an eigen value of A^{-1}.

5. a) Show that the characteristic polynomial of any diagonalizable linear operator splits. **(4+4+8)**

b) State and prove the Cauchy-Schwarz inequality.

c) Let V be an inner product space and $u, v \in V$ then, prove the following

i) $||u + v||^2 - ||u - v||^2 = 4(u,v)$

ii) $\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$

6. a) Define a normal operator. Prove that on operator T is normal if and only if there exists an orthonormal basis of V, a finite dimensional inner product space consisting of eigen vectors of T. **(6+5+5)**

b) Let V be a finite - dimensional inner product space and let T be a linear operator on V. Then show that there exists a unique function $T^*: V \to V$ *such that* $\left(T(x), y\right) = \left(x, T^*(y)\right)$ for all $x, y \in V$. Further more, show that T* is linear.

c) Let B be a bilinear form on a finite dimensional vector space V and let B be an ordered basis for V. Then B is symmetric if and only if $\psi_B(B)$ is symmetric.

7. a) If $T \in A(V)$ has all its characteristic roots in F, then show that there is a basis of V in which the matrix of T is triangular. **(6+5+5)**

b) Let $C[-\pi, \pi]$ be the inner product space of all continuous functions defined on $[-\pi, \pi]$ with the inner product defined by $< f, g >= \int_{-\pi}^{\pi} f(t)g(t)\,dt$, then prove that sint and cost are orthogonal functions in $C[-\pi, \pi]$.

c) If u = (-1,4,-3) be a vector in the inner product space with the standard inner product, then find a basis of the subspace $u^{\perp}$ *of* R^3.

8. If $W \subset V$ is invariant under T, then T induces a linear transformation $\overline{T}$ *on* V/W defined by $\overline{T}(v+W) = T(v)+W$. If T satisfies the polynomial $q(x) \in F[x]$, then so does $\overline{T}$. If $p_1(x)$ is the minimal polynomial for $\overline{T}$ over F and if $p_1(x)$ is that for T, then $p_1(x)/p(x)$. **(16)**
