MMDSC-3.3

Sl.No. :

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III Semester M.Sc. Degree Examination, June - 2025
(CBCS, SLM Scheme)
MATHEMATICS
Course - III : Functional Analysis - I

Time : 3 Hours **Max. Marks : 80**

Instruction : Answer any five full questions.

Q1) a) Define complete metric space. If $X = \mathbb{R}^n$ and $d\left((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\right) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$ then show that (X, *d*) is a metric space.

b) State and prove Picard's Theorem.

[8+8]

Q2) a) Let (X, *d*) be a complete and totally bounded metric space, then show that X is compact.

b) Let (X, *d*) be a metric space, then show that X is sequentially compact if and only if X has Balzano-Weierstrass property.

[8+8]

Q3) a) Let $T : N \to N'$ be a linear operator from a normed linear space N into a normed linear space N', then show that T is a continuous linear operator and bounded linear operator.

b) Show that a normed linear space X is complete if and only if every absolutely convergent series in X is convergent.

[8+8]

Q4) a) State and prove Hahn-Banach theorem for any real linear space.

b) State and prove Stone-Weierstrass theorem for complex case.

[10+6]

Q5) a) Show that l_∞ is a Banach space.

b) If $T : X \to Y$ is a one to one, continuous linear transformation from X onto Y, then show that T is a homomorphism from a Banach space X onto Banach space Y.

[8+8]

Q6) a) If $\phi : X \to Y$ is a one to one continuous linear mapping from a Banach space X onto a Banach space Y, then show that $\phi^{-1} : Y \to X$ is also one to one continuous linear mapping from Y onto X.

b) Let N be a normed linear space, then show that non empty subset X of N is bounded if and only if $f(x)$ is a bounded set of numbers for every $f \in N^*$, where N^* is the conjugate of N.

[8+8]

Q7) a) Let H be a Hilbert space, then show that $\mu < x, y >= \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$.

b) Show that $L^2(I) = \left\{ f : I \to \mathbb{R} \text{ (OR) } \mathbb{C} : \int_a^b |f|^2 \, dx < \infty \right\}$ is a Hilbert space.

[8+8]

Q8) a) Let M be a closed linear subspace of a Hilbert space H, then show that $H = M \oplus M^{\perp}$.

b) Let M be a closed proper subspace of a normed linear space N and $\in > 0$, then show that there is a point $x_\in$ on the unit sphere such that

$$\inf \{ \|x_0 - x_\in\| : x_0 \in M \} > 1 - \in .$$

[8+8]

