# MMDSC-4.3

Sl.No. :

**Total No. of Pages : 2**

## Fourth Semester M.Sc. Degree Examination, November - 2024

### (SLM Scheme) (CBCS)

### MATHEMATICS

### Lattice Theory

**Time : 3 Hours** **Max. Marks : 80**

**Instructions : 1) Answer any five full questions.**
**2) All questions carry equal marks.**
**3) Use of scientific calculator is permitted.**

***Q1)*** a) Let $n \in \mathbb{N}$. Let $D_n$ Denote the set of all positive divisors of $n$. Then $< D_n, | >$ is a subposet of $< \mathbb{N}, | >$. Show that $< D_n, | >$ is self dual for any $n \in \mathbb{N}$.

b) Show that the notions of minimal elements and minimum coincide for chains.

c) Let A be any set. Show that $(P(A), \subseteq)$ is a chain if and only if $|A| \leq 1$.

**[6+6+4]**

***Q2)*** a) State Hausdorff's Maximal Chain Principle and Zorn's Lemma. Prove that the Zorn's Lemma equivalent to Hausdorff's Maximal Chain Principle.

b) Define the length of a poset. Is every poset of finite length a finite poset? What about the converse? Justify.

**[12+4]**

***Q3)*** a) Give set theoretic and algebraic definitions of a lattice and prove their equivalence.

b) Let $(P, \leq)$ be a poset in which every pair of elements have a *glb*. Define a binary operation $\wedge$ on $P$ as follows. For $a, b \in P$, define $a \wedge b = glb\{a, b\}$. Show that $(P, \wedge)$ is a semilattice.

c) Show that union of two ideals is an ideal if and only if one is contained in the other.

**[8+4+4]**

***Q4)*** a) Show that an ideal $I$ of a lattice $L$ is a prime ideal if and only if $L - I$ is a prime dual ideal.

b) Prove that the set union of members of any ideal chain of a lattice is again an ideal of $L$.

c) Show that every ideal of a lattice is principal if and only if $L$ satisfies acc.

**[5+5+6]**

***Q5)*** a) Define a conditionally complete lattice. Prove that a lattice $L$ is conditionally complete if and only if every non-empty subset of $L$ bounded above in $L$ has a join.

b) Define a compact element, compactly generated lattice. Prove that the subset lattice of any set is compactly generated.

**[8+8]**

***Q6)*** a) Prove that any homomorphic image of a distributive lattice is distributive.

b) Let $L$ be any lattice and $I(L)$ be its ideal lattice. Prove that a lattice $L$ is distributive if and only if $I(L)$ is distributive.

c) State and prove isomorphism theorem for modular lattices.

**[5+5+6]**

***Q7)*** a) State and prove Dedekind's Modularity Criteria.

b) Prove that the following statements are equivalent for any lattice $L$.

i) $L$ is modular.

ii) For $x, y, z \in L$, $y \geq z$, $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ imply $y = z$.

iii) No interval $[a, b]$ of $L$ contains an element having 2 different comparable relative complements in $[a, b]$.

**[10+6]**

***Q8)*** a) Prove that every complemented modular lattice is relatively complemented.

b) Prove that every complete Boolean algebra is infinitely distributive.

c) Define a Boolean Ring. Prove that any Boolean ring can be made into a Boolean Algebra by suitably defining the Boolean operations.

**[4+4+8]**

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