MMDSC-4.3

Sl.No. :

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Fourth Semester M.Sc. Degree Examination, October - 2025
(SLM Scheme) (CBCS)
MATHEMATICS
Lattice Theory

Time : 3 Hours **Max. Marks : 80**

Instructions : 1) Answer any five full questions.
2) All questions carry equal marks.
3) Use of scientific calculator is permitted.

Q1) a) Let D_n denote the set of all positive divisors of $n \in \mathbb{N}$. Then show that $< D_n, | >$ is a subposet of $< \mathbb{N}, | >$. Further prove that $< D_n, | >$ is self dual for any $n \in \mathbb{N}$.

b) Define the notions of lower bound, greatest lower bound, minimum element and minimal element. Is a minimal always a minimum element? What about the converse? Justify.

c) Draw the diagram of D_{210} under "divides" relation.

[6 + 6 + 4]

Q2) a) State Hausdorff's Maximal Chain Principle and Zorn's Lemma. Show that Hausdorff's maximal Chain Principle implies Zorn's Lemma.

b) Define the length of a poset. Is every poset of finite length a finite poset? What about the converse? Justify.

c) Show that any poset P is order isomorphic with a collection of subsets of P ordered by inclusion.

[6 + 5 + 5]

Q3) a) Give set theoretic and algebraic definitions of a lattice and prove their equivalence.

b) Show that a non-empty intersection of an ideal and a dual ideal in a lattice is a convex sublattice. Conversely prove that every convex sublattice can be uniquely expressed in this form.

[10 + 6]

Q4) a) Prove that a one to one map f of a lattice L_1 onto a lattice L_2 is an isomorphism if and only if f and f^{-1} are order preserving.

b) Let L be a lattice. Show that $I(L)$ has minimum element if and only if L has minimum element.

c) Define a congruence relation on a lattice L and prove that every congruence class is a convex sublattice of L.

[8 + 4 + 4]

Q5) a) Define a conditionally complete lattice. Prove that a lattice L is conditionally complete if and only if every non-empty subset of L bounded above in L has a join.

b) Prove that every homomorphic image of a modular lattice is modular.

c) In any lattice L, prove that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $\forall\ x,y,z \in L$ if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $\forall\ x,y,z \in L$.

[8 + 4 + 4]

Q6) a) State and prove Birkhoff's distributivity criterion.

b) Prove that lattice L is distributive if and only if med (x,y,z) exists for every triplet x,y,z of elements of L. **[8 + 8]**

Q7) a) State and prove prime ideal theorem.

b) State and prove isomorphism theorem for modular lattices.

c) Prove that a lattice L is modular if and only if $I(L)$ is modular.

[8 + 4 + 4]

Q8) a) Show that a Boolean algebra B is complete and atomic if and only if B is isomorphic with the subset lattice of some set.

b) Define a Boolean Ring. Prove that any Boolean ring can be made into a Boolean Algebra by suitably defining the Boolean operations.

[8 + 8]

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