MMDSE-4.6

Sl.No. :

Total No. of Pages : 3

Fourth Semester M.Sc. Degree Examination, October - 2025
(SLM Scheme) (CBCS)
MATHEMATICS
Differential Geometry

Time : 3 Hours **Max. Marks : 80**

Instructions : ***1) Answer any five questions***

2) All questions carry equal marks

Q1) a) Define the following with an example for each:

i) a level curve ii) a parametrized curve

Obtain the Parametrized curve for the following: I) Astroid II) Limacon.

b) Show that a parametrized curve has a unit-speed reparametrization if and only if it is regular. Further, show that the logarithmic spiral $\gamma(t) = (e^t \cos t, e^t \sin t)$ has a unit-speed reparametrization.

[8 + 8]

Q2) a) Let $\gamma(s)$ be a unit-speed plane curve, and let $\varphi(s)$ be a turning angle for γ. Then prove that $K_s = \frac{d\varphi}{ds}$. Further, show that any regular plane curve γ whose curvature is a positive constant is part of a circle.

b) Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then prove that $\tau = \frac{(\gamma' \times \gamma'') \bullet \gamma'''}{\|\gamma' \times \gamma''\|^2}$.

[8 + 8]

Q3) a) Let $F:[0,\pi]\to\mathbb{R}$ be a smooth function such that $F(0)=F(\pi)=0$. Then show that $\int_0^{\pi}\left(\frac{dF}{dt}\right)^2 dt \geq \int_0^{\pi} F(t)^2 dt$ and equality holds if and only if $F(t)=a\sin t$ for all $t\in[0,\pi]$, where a is a constant.

b) Define: i) surface patch ii) smooth surface. Further, prove the following:
I) a unit 2-sphere S^2 is a smooth surface.

II) Let $f_1:S_1\to S_2$ be a diffeomorphism. If σ_1 is a surface patch on S_1, then $f\bullet\sigma_1$ is a surface patch on S_2.

[8 + 8]

Q4) a) Let S and $\tilde{S}$ be a smooth map. Then show that f is a local diffeomorphism if and only if, for all $p\in S$, the linear map $D_p f:T_pS\to T_{f(p)}\tilde{S}$ is invertible.

b) Define the first fundamental form of a surface. Obtain the first fundamental form of the following:

i) $\sigma(u,v)=(u,v,\sqrt{1-u^2-v^2})$

ii) $\sigma(u,v)=((a+b\cos(u))\cos(v),a+b\cos(u)\sin(v),b\sin(u))$

[8 + 8]

Q5) a) Define an isometry. Let ρ and $\tilde{\rho}$ are the intrinsic distance functions of S_1 and S_2, repectively. Prove that isometries preserve intrinsic distances if $f:S_1\to S_2$ is an isometry of surfaces.

b) Show that a diffeomorphism $f:S_1\to S_2$ is conformal if and only if, for any surface patch σ of S_1, the first fundamental foms of the patches σ of S_1 and $f\bullet\sigma$ of S_2 are proportional.

[8 + 8]

Q6) a) Let p and q be distinct points of S^2. If $p \neq q$, then show that the short great circle arc joining p and q is the unique curve of shortest length joining p and q.

b) Define: i) the Gauss map, ii) the Weingarten map. If γ is a unit-speed curve on an oriented surface patch σ, then show that its normal curvature is given by $K_n = Ldu^2 + 2Mdudv + Ndv^2$, where $Ldu^2 + 2Mdudv + Ndv^2$ is the second fundamental form of σ.

[8 + 8]

Q7) a) Let K_1 and K_2 be the principal curvatures at a point p of a surface patch σ. Then prove the following:

i) If $K_1=K_2=K$ then $F_{11} = KF_1$ and hence every tangent vector' to σ at p is a principal vector.

ii) if $K_1 \neq K_2$ then any two non-zero principal vectors t_1 and t_2 corresponding to K_1 and K_2, respectively, are perpendicular.

b) Obtain the Gaussian curvature for the surface of revolution:
$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$, where $f > 0$ and $f'^2 + g'^2 = 1$ everywhere.

[8 + 8]

Q8) a) Let p be a point of a flat surface S, and assume that p is not an umbilic. Then show that there is a patch of S containing p that is a ruled surface.

b) Let $\sigma : U \to R^3$ be a surface, let $(u_0,v_0) \in U$, and Let $\delta > 0$ be such that the closed disc $R_\delta = \{(u,v) \in R^3 \mid (u-u_0)^2 + (v-v_0)^2 \leq \delta^2\}$ with center (u_0,v_0) and radius δ is contained in U. Then prove that $\lim_{\delta \to 0} \frac{A_N(R_\delta)}{A_\sigma(R_\delta)} = |K|$ where K is the Gaussian curvature of σ at $\sigma(u_0,v_0)$.

[8 + 8]