**DECEMBER 2024** **U/ID 32516/UCM6A**

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Define subspace.

2. Define linear combination.

3. Show that the vectors (1,0,0), (0,1,0) and (0,0,1) are linearly independent.

4. Define dual space.

5. If $\dim_F V = 6$, find $\dim_F Hom(V,V)$.

6. Define norm of inner product space.

7. State Schwarz Inequality.

8. Prove that $W^{\perp}$ is a subspace of $V$.

9. Define Algebra.

10. Define rank of transformation.

11. Define matrix.

12. Define invariant subspace.

PART B — ($5 \times 5 = 25$ marks)

Answer any FIVE questions.

13. If $V$ is a vector space over $F$, prove that

(a) $\alpha 0 = 0$ for $\alpha \in F$

(b) $0v = 0$ for $v \in V$

(c) $(-\alpha)v = -(\alpha v)$ for $\alpha \in F$, $v \in V$.

14. Prove that $F^{(n)}$ is isomorphic if and only if $n = m$.

15. If $V$ is finite-dimensional, prove that $\psi$ is an isomorphism of $V$ onto $\hat{\hat{V}}$.

16. If $V$ is a finite-dimensional inner product space and $W$ is a subspace of $V$, prove that $\left(W^{\perp}\right)^{\perp} = W$.

17. If $u, v \in V$ and $\alpha, \beta \in F$, prove that $(\alpha u + \beta v, \alpha u + \beta v) = \alpha\overline{\alpha}\langle u, v\rangle + \alpha\overline{\beta}\langle u, v\rangle + \overline{\alpha}\beta\langle u, v\rangle + \beta\overline{\beta}\langle u, v\rangle$.

18. If $V$ is finite – dimensional over $F$ and if $T \in A(V)$ is singular, prove that there exists an $S \neq 0$ in $A(V)$ such that $ST = TS = 0$.

19. If $W \subset V$ is invariant under $T$, prove that $T$ induces a linear transformation $\overline{T}$ on $v/w$, defined by $(v + w)T = vT + wT$. Also prove that if $T$ satisfies, the polynomial $q(x) \in F[x]$ then so does $\overline{T}$.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. If $S$ and $T$ are subsets of $V$, prove that

(a) $S \subset T$ implies $L(S) \subset L(T)$

(b) $L[S \cup T] = L(S) + L(T)$

(c) $L(L(S)) = L(S)$.

21. If $V$ and $W$ are of dimensions $m$ and $n$, respectively over $F$, prove that $Hon(V, W)$ is of dimension on $mn$ over $F$.

22. State and prove Gram – Schmidt orthogonalization process.

23. If $V$ is finite dimensional over $F$, prove that for $S, T \in A(V)$

(a) $r(ST) \leq r(T)$

(b) $r(TS) \leq r(T)$ also $(r \subset ST) \leq \min\{r(T), r(S)\}$

(c) $r(ST) = r(TS) = r(T)$ for $S$ regular in $A(V)$.

24. If $V$ is $n$-dimensional over $F$ and if $T \in A(V)$ has all its characteristic roots in $F$, prove that $T$ satisfies a polynomial of degree $n$ over $F$.

———————