MAY 2023 **U/ID 32516/UCM6A**

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. If V is a vector space over F, then prove that $\alpha 0 = 0$ for all $\alpha \in F$.

2. Define linear span of a set S.

3. Prove that the vectors $(1,1,1,1),(1,1,1,0),(0,0,1,0)$ and $(1,1,0,0)$ are linearly independent in $\mathbb{R}^4$.

4. Define linear functional on V.

5. Find second dual of a vector space V over F.

6. Prove that $(v, \alpha u + \beta w) = \overline{\alpha}(v,u) + \overline{\beta}(u,w)$ for $u,v,w \in V$, $\alpha\beta \in F$ where V is an inner product space on F.

7. When do you say that u is orthogonal to v?

8. Prove that $\{0\}^{\perp} = V$ where V is an inner product space.

9. If $T \in A(V)$ is regular, then find rank (T).

10. If $x^2 - 3x + 2$ is the minimal polynomial of $T \in A(V)$, then find its characteristics roots.

11. Define matrix of $T \in A(V)$ in the basis $\{v_1, ..., v_n\}$.

12. When do you say that the matrix $A \in F_n$ is similar to triangular matrix?

PART B — (5 × 5 = 25 marks)

Answer any FIVE questions.

13. If V is a vector space over F and W is its subspace then prove that V/W is a vector space.

14. If W is subspace of V, then prove that $A(W)$ is a subspace of $\hat{V}$.

15. Prove that $(W^{\perp})^{\perp} = W$.

16. Prove that every orthonormal set in an innerproduct space is linearly independent.

17. If $T, S \in A(V)$ and V is finite dimensional, then prove that $r(TS) \leq r(T)$.

18. Prove that $\lambda \in F$ is a characteristic root of $T \in A(V)$ if and only if $T(v) = \lambda v$, for some $v \neq 0 \in V$.

19. If V is the vector space of polynomials of degree 3 or less over F and T is defined by $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_0 + \alpha_1 (x+1) + \alpha_2 (x+1)^2 + \alpha_3 (x+1)$ then find matrix of T in the basis $\{1, x, x^2, x^3\}$.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. If W is a subspace of a finite dimensional space V, then prove that $\dim \frac{V}{W} = \dim V - \dim W$.

21. If V and W are vector spaces over F of dimensions m and n respectively, then prove that $Hom(V, W)$ is of dimension of mn.

22. If W is a subspace of a finite dimensional inner product space V, then prove that $V = W \oplus W^{\perp}$.

23. If V is finite dimensional over F, then prove that $T \in A(V)$ is regular if and only if T maps V onto V.

24. If V is n-dimensional over F and if $m_1(T)$ and $m_2(T)$ one the matrices of $T \in A(V)$ in the base $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ respectively, then prove that there is $C \in F_n$ such that $m_2(T) = Cm_1(T)\,C^{-1}$.