MAY 2025 **U/ID 32516/UCM6A**

Time : Three hours Maximum : 75 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Define vector space Homomorphism.

2. If V is a vector space over F, prove that $\alpha 0 = 0$ for $\alpha \in F$.

3. Define linear dependent.

4. If $\dim_F V = n$, find $\dim_F Hom(V, V)$ and $\dim_F Hom(V, F)$.

5. Define inner product space.

6. Define orthogonal complement of a subspace.

7. State Gram Schmidt orthogonalization process.

8. What is range of linear transformation?

9. Define characteristic root.

10. If V is finite dimensional over F and it $T \in A(V)$ is right invertible, prove that it is invertible.

11. Compute $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}^2$.

12. Prove that $M(ST) = M(S)\, M(T)$.

PART B — (5 × 5 = 25 marks)

Answer any FIVE questions.

13. If $V_1 ..... V_n \in V$ are linearly independent, prove that every element in their linear span has a unique representation in the form $\lambda v_1 + ... + \lambda_n v_n$ with the $\lambda_i \in F$.

14. Prove that $L(S)$ is a subspace of V.

15. Prove that $Hom(V, W)$ is a vector space.

16. State and prove Schwarz inequality.

17. If $\{v_i\}$ is an orthonormal set, prove that the vectors in $\{v_i\}$ are linearly independent. Also prove that if $w = \alpha_1 v_1 + ... + \alpha_n v_n$ then $\alpha_i = \langle w, v_i \rangle$ for $i = 1, 2, .. n$.

18. Prove that if V is finite-dimensional over F, then $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0.

19. Prove that if V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, then T satisfies a polynomial of degree n over F.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Prove that if A and B are finite dimensional subspace of a vector space V, then $A+B$ is finite-dimensional and

$$\dim(A+B) = \dim A + \dim B - \dim(A \cap B).$$

21. If V and W are of dimensions m and n respectively, over F, prove that $Hom(V, W)$ is of dimension mn over F.

22. Prove that if V is a finite-dimensional inner product space and if W is a subspace of V, then $V = W + W^{\perp}$. More particularly, V is the direct sum of W and $W^{\perp}$.

23. Prove that if V is finite-dimensional over F, then $T \in A(V)$ is regular if and only if T maps V onto V.

24. Prove that if V is n-dimensional over F and if $T \in A(V)$ has the matrix $m_1(T)$ in the basis $v_1, \ldots v_n$ and the matrix $m_2(T)$ in the basis $w_1 \ldots w_n$ of V over F, then there is an element $C \in F_n$ such that $m_2(T) = C m_1(T) C^{-1}$.
