DECEMBER 2023 **P/ID 37502/PMA1B**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN the questions.

1. Verify whether $f(x) = \begin{cases} x^2 \cos(1/x), & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ is a functions of bounded variation on $[0,1]$.
2. Give an example of conditionally convergent series.
3. Define upper and lower Stieltje's integrals.
4. State the Euler's summation formula.
5. Define upper and lower Stieltjes sum of a function f with respect to α.
6. Write the sufficient condition for the existence of Riemann Stieltjes integrals.
7. Define a set of measure zero.
8. Given $f(p,q) = \dfrac{pq}{p^2+q^2}$ $(p = 1,2,...,q = 1,2,...)$. Find $\lim\limits_{p\to\infty}\left(\lim\limits_{q\to\infty} f(p,q)\right)$ and $\lim\limits_{p,q\to\infty} f(p,q)$.
9. State Arzela's theorem.
10. State Weierstrass M-test.
11. Define uniform convergence of a sequence of functions.
12. Define convergence in mean.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let f be an increasing function defined on $[a,b]$ and let $x_0, x_1, ..., x_n$ be $n+1$ points such that $a = x_0 < x_1 < x_2 < ... < x_n = b$. The prove that

$$\sum_{k=1}^{n-1}[f(x_k+) - f(x_k-)] \le f(b) - f(a).$$

14. Assume that α is increasing on [a,b]. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$ and if $f(x) \le g(x)$ for all x in $[a,b]$, then prove that

$$\int_a^b f(x)d\alpha(x) \le \int_a^b g(x)d\alpha(x).$$

15. Let f be defined and bounded on $[a,b]$ and let $\varepsilon > 0$ be given. Assume that $w_f(x) < \varepsilon$ for every x in $[a,b]$. Then prove that there exists a $\delta > 0$ (depending on ε) such that for every closed subinterval $T \subseteq [a,b]$, we have $\Omega_f(T) <\in$ whenever the length of T is less than δ.

16. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series with sums A and B, respectively. Let f be a double sequence defined by the equation

$f(m,n) = a_m b_n$, if $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$.

The prove that $\sum_{m,n} f(m,n)$ converges absolutely and has the sum AB.

17. Assume that $\lim_{n\to\infty} f_n = f$ on $[a,b]$. If $g \in R$ on $[a,b]$, define $h(x) = \int_a^x f(t)g(t)dt$, $h_n(x) = \int_a^x f_n(t)g(t)dt$, if $x \in [a,b]$. Then prove that $h_n \to h$ uniformly on $[a,b]$.

18. Prove that absolute convergence of $\sum a_n$ implies convergence.

19. Assume that $\sum a_n(z-z_0)^n$ converges for each $z \in B(z_0;r)$. Then prove that the function f defined by the equation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{if } z \in B(0;r),$$

has a derivative $f'(z)$ for each z in $B(z_0;r)$, given by $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Let $\sum a_n$ be absolutely convergent series with real valued terms. Let x and y be given numbers in the closed interval $[-\infty,\infty]$, with $x \le y$. Prove that there exists a rearrangement $\sum b_n$ of $\sum a_n$ such that $\liminf_{n\to\infty} t_n = x$ and $\limsup_{n\to\infty} t_n = y$, where $t_n = b_1 + \ldots + b_n$.

21. Assume that $f \in R(\alpha)$ on $[a,b]$ and assume that α has a continuous derivative α' on $[a,b]$. Then prove that the Riemann-integral $\int_a^b f(x)\alpha'(x)dx$ exists and we have $\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$.

22. State and prove the first Mean-Value theorem for Riemann-Stieltjes integrals.

23. State and prove Abel's limit theorem.

24. Let α be of bounded variation on $[a,b]$. Assume that each term of the sequence $\{f_n\}$ is a real-valued function such that $f_n \in R(\alpha)$ on $[a,b]$ for each $n = 1,2,...$ Assume that $f_n \to f$ uniformly on $[a,b]$ and define $g_n(x) = \int_a^x f_n(t)d\alpha(t)$ if $x \in [a,b]$, $n = 1,2,....$ Then prove that

    (a) $f \in R(\alpha)$ on $[a,b]$.

    (b) $g_n \to g$ uniformly on $[a,b]$ where $g(x) = \int_a^x f(t)d\alpha(t)$.
