DECEMBER 2024 **P/ID 37502/PMA1B**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Define Step function.
2. Define absolute and conditional convergence.
3. Write the additive property of total variation.
4. Define Oscillation of f on T.
5. State the theorem on reduction of a Riemann-Stieltjes integral to a finite sum.
6. State second Mean-Value theorem for Riemann-Stieltjes integrals.
7. Define Cesáro summable.
8. Write the Cauchy condition for products.
9. State Merten's theorem.
10. Write the Cauchy condition for uniform convergence of series.
11. State the Lebesgue criterion for Riemann integrability.
12. State Dirichlet's test for uniform convergence of a series.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let f be of bounded variation on $[a,b]$. Let V be defined on $[a,b]$ as follows :

$V(x) = V_f(a, x)$ if $a < x \le b$, $V(a) = 0$. Then prove that

(a) V is an increasing function on $[a,b]$

(b) $V - f$ is an increasing function on $[a,b]$.

14. Let $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$, prove that $c_1 f + c_2 g \in R(\alpha)$ on $[a,b]$, for any two constants c_1 and c_2. Also prove that

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha .$$

15. Let F be a countable collection of sets in $\mathbb{R}$, say $F = \{F_1, F_2, \ldots\}$, each of which has measure zero. Prove that their union

$$S = \bigcup_{k=1}^{\infty} F_k$$

also has measure zero.

16. Assume that $\lim_{p,\, q\to\infty} f(p, q) = a$. For each fixed p, assume that the limit $\lim_{q\to\infty} f(p, q)$ exists. Then prove that the limit $\lim_{p\to\infty}\left(\lim_{q\to\infty} f(p, q)\right)$ also exists and has the value a.

17. Assume that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ on $[a,b]$. Define

$$h(x) = \int_a^x f(t)g(t)dt,\ h_n(x) = \int_a^x f_n(t)g(t)dt,$$

if $x \in [a,b]$. Then prove that $h_n \to h$ uniformly on $[a,b]$.

18. Let f be a double sequence and let $\mathbb{Z}^+$ denote the set of positive integers. For each $n = 1, 2, \ldots$, define a function g on $\mathbb{Z}^+$ as follows :

$g_m(m) = f(m, n)$, if $m \in \mathbb{Z}^+$

Assume that $g_n \to g$ uniformly on $\mathbb{Z}^+$, where $g(m) = \lim_{n\to\infty} f(m, n)$. If the iterated limit $\lim_{n\to\infty}(\lim_{n\to\infty} f(m, n))$ exists, then prove that the double limit $\lim_{m,\, n\to\infty} f(m, n)$ also exists and has the same value.

19. Assume that $\alpha\nearrow$ on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then prove that $f^2 \in R(\alpha)$ on $[a,b]$.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. Let $\sum a_n$ be an absolutely convergent series having sum s. Then every rearrangement of $\sum a_n$ also converges absolutely and has sum s.

21. Assume that $\alpha \nearrow$ on $[a,b]$. Then prove that the following are equivalent :

    (a) $f \in R(\alpha)$ on $[a,b]$

    (b) f satisfies Riemann's condition with respect to α on $[a,b]$.

    (c) $\underline{I}(f, \alpha) = \overline{I}(f, \alpha)$.

22. Assume that α is of bounded variation on $[a,b]$. Let $V(x)$ denote the total variation of α on $[\alpha, x]$ if $a < x \leq b$ and let $V(a) = 0$. Let f be defined and bounded on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$ then prove that $f \in R(V)$ on $[a,b]$.

23. State and prove Bernstein's theorem.

24. State and prove the Cauchy condition for uniform convergence of sequence of functions.
