(6 pages)

**MAY 2025** **P/ID 37502/PMA1B**

Time : Three hours Maximum : 80 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Verify whether $f(x) = \begin{cases} x^2 \cos(1/x), & if \ x \neq 0 \\ 0, & if \ x = 0 \end{cases}$ is a functions of bounded variation on [0, 1].

2. Define absolute and conditional convergence.

3. Write the additive property of total variation.

4. State the Euler's summation formula.

5. State the sufficient condition for the existence of Riemann Stieltjes integrals.

6. State second Mean-Value theorem for Riemann Stieltjes integrals.

7. Define a set of measure zero.

8. State Tauber's theorem.

9. State Merten's theorem.

10. State Weierstrass M-test.

11. Define Pointwise convergence of sequence of functions.

12. State Dirichlet's test for uniform convergence of a series.

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. Let $f$ be an increasing function defined on $[a, b]$ and let $x_0, x_1 .. x_n$ be $n+1$ points such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$. Then prove that

$$\sum_{k=1}^{n-1} \left[ f(x_k +) - f(x_k -) \leq f(b) - f(a) \right]$$

14. Assume that $\alpha$ is $\nearrow$ on $[a, b]$. Then prove that $\underline{I}(f, \alpha) = \bar{I}(f, \alpha)$.

15. Let $F$ be a countable collection of sets in $\mathbb{R}$, say $F = \{F_1, F_2, \ldots\}$, each of which has measure zero. Prove that their union $S = \bigcup_{k=1}^{\infty} F_k$ also has measure zero.

16. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series with sums $A$ and $B$, respectively. Let $f$ be a double sequence defined by the equation

$$f(m, n) = a_m b_n, \text{ if } (m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$$

Then prove that $\sum_{m, n} f(m, n)$ converges absolutely and has the sum $AB$.

17. Assume that $f_n \to f$ uniformly on $S$. If each $F_n$ is continuous at a point $c$ of $S$, then prove that the limit function $f$ is also constant at $c$.

18. Let $f$ be a double sequence and let $\mathbb{Z}^+$ denote the set of positive integers. For each $n$ = 1,2...., define a function $g$ on $\mathbb{Z}^+$ as follows:

$$g_m(m) = f(m,. n), \text{ if } m \in \mathbb{Z}^+$$

Assume that $g_n \to g$ uniformly on $\mathbb{Z}^+$ where $g(m) = \lim_{n\to\infty} f(m, n)$. If the iterated limit $\lim_{n\to\infty} (\lim_{n\to\infty} f(m, n))$ exists, then prove that the double limit $\lim_{m,\, n\to\infty} f(m, n)$ also exists and has the same value.

19. Assume that $\sum a_n(z - z_0)^n$ converges for each $z \in B(z_0; r)$. Then prove that the function $f$ defined by the equation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ if } o \in B(0; r)$$

has a derivative $f'(z)$ for each $z$ in $B(0; r)$, given by

$$f'(z) = \sum_{n=0}^{\infty} n\ a_n\ z^{n-1}$$

PART C — (3 × 10 = 30 marks)

Answer any THREE questions

20. Let $f$ be of bounded variation on [a, $b$]. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Prove that every point of continuity of $f$ is also a point of continuity of $V$. Prove that the converse is also true.

21. Assume that α is $\nearrow$ on [$a$, $b$]. Then prove that the following are equivalent:

(a) $f \in R(\alpha)$ on [$a$, $b$]

(b) $f$ satisfies Riemann's condition with respect to $\alpha$ on [$a$, $b$]

(c) $\underline{I}(f, \alpha) = \overline{I}(f, \alpha)$.

22. State and prove the first Mean-Value theorem for Riemann-Stieltjes integrals.

23. Assume that $\sum a_n(z-z_0)^n$ converges in $z \in B(z_0; r)$. Suppose that the equation

$$f(z) = \sum_{n=0}^{\infty} a_0(z-z_0)^n$$

is known to be valid for each $z$ in some open subset $S$ of $B(z_0; r)$. For each point $z_1$ in $S$, prove that there exists a neighborhood $B(z_1; R) \subseteq S$ in which $f$ has a power series expansion of the form

$$f(z) = \sum_{k=0}^{\infty} b_k (z-z_1)^n ,$$

where $b_k = \binom{n}{k} a_n (z_1 - z_0)^{n-k}$, $(k = 0, 1, 2, \ldots)$

24. State and prove the Cauchy condition for uniform convergence of sequence of functions.

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