DECEMBER 2024 **P/ID 37503/PMA1C**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Solve $y'' + w^2 y = 0$.

2. State Uniqueness theorem for linear equations with constant coefficients.

3. Check whether the sets of functions $\phi_1(x) = x, \phi_2(x) = e^{2x}, \phi_3(x) = |x|$ on $-\infty < x < \infty$ are linearly independent or dependent.

4. Define Wronskian.

5. State Existence theorem for linear equations with variable coefficients.

6. Define Gramian.

7. Compute the indicial polynomial and its roots of $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$.

8. Write Bessel function of zero order of the second kind.

9. Find all real valued solutions of the equation $y' = x^2 y$.

10. Write the Lipschitz condition.

11. Write the Euler equation.

12. Define singular points.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let ϕ_1, ϕ_2 be two solutions of $L(y) = 0$ on an interval I and let x_0 be any point in I. Prove that ϕ_1, ϕ_2 are linearly independent on I if and only if $W(\phi_1, \phi_2)(x_0) \neq 0$.

14. Solve $y'' - y' - 2y = e^{-x}$.

15. Let ϕ be any solution of $L(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$ on an interval I containing a point x_0. For $k = 1 + |a_1| + \ldots + |a_n|$ and for all x in I, prove that

$$\|\phi(x_0)\| e^{-k|x-x_0|} \leqq \|\phi(x)\| \leqq \|\phi(x_0)\| e^{k|x-x_0|}$$

16. Verify that the function $\phi(x) = x^3$ satisfies $x^2y'' - 7xy' + 15y = 0, (x > 0)$. Also find the second independent solution of the equation.

17. Compute the indicial polynomials and their roots for the equation

$$x^2y'' + (\sin x)\, y' + (\cos x)\, y = 0\,.$$

18. Determine whether the equation $(x + y)dx + (x - y)\, dx = 0$ is exact and solve it.

19. Show that the equation
$f(x, y) = x^2 \cos^2 y + y \sin^2 x$, on
$S : |x| \leqq 1, |y| < \infty$ satisfy Lipschitz conditions.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. State and prove the Existence theorem.

21. Suppose the constants $a_1, \ldots, a_n$ in the equation $L(y) = y^{(n)} + a_1\, y^{(n-1)} + \ldots + a_n\, y = 0$ are all real. Prove that there exists a set of n linearly independent real valued solutions and every real valued solution is a linear combination of these with real coefficients. If a solution satisfies real initial conditions, it is real valued.

22. Prove that $\phi_1, \ldots, \phi_n$ are n solutions of $L(y) = 0$ on an interval I they are linearly independent there if and only if $W(\phi_1, \ldots, \phi_n)(x) \neq 0$ for all x in I.

23. Let ϕ be any solution for $x > 0$ of the Bessel equation of order α $x^2 y'' + xy' + (x^2 - \alpha^2)\, y = 0$ and put $\psi(x) = x^{1/2}\, \phi(x)$. Show that ψ satisfies the equation $y'' + \left[1 + \dfrac{\frac{1}{4} - \alpha^2}{x^2}\right] y = 0$ for $x > 0$.

24. Let M, N be two real valued functions which have continuous first partial derivatives on some rectangle $R : |x - x_0| \leqq a, |y - y_0| \leqq b$ .Prove that the equation $M(x, y) + N(x, y)\, y' = 0$ is exact in R if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ in R.
