MAY 2023 **P/ID 37503/PMA1C**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Find all the solutions of the equation $y'' + 16y = 0$.

2. Determine whether the functions $\phi_1 = x$, $\phi_2 = e^{rx}$ are linearly dependent or independent.

3. Solve: $y'' + 4y = \cos x$.

4. Define Wronskian.

5. State Uniqueness theorem for linear equations with variable coefficients.

6. Define a singular point.

7. Write the Bessel equation.

8. Write the Legendre equation.

9. Find the singular point of $x^2y'' + (x + x^2)y' - y = 0$.

10. Define Frobenius Method.

11. Define a basis.

12. Define an exact differential equation.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let ϕ_1, ϕ_2 be two solutions of $L(y) = 0$ on an interval I and x_0 be any point on I. Prove that ϕ_1, ϕ_2 are linearly independent on I if and only if $W(\phi_1, \phi_2)(x_0) \neq 0$.

14. Let ϕ be any solution of $L(y) = y'' + a_1 y' + a_2 y = 0$ on an interval I containing x_0. Prove that for all x in I,

$\|\phi(x_0)\| e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{k|x-x_0|}$ where

$\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2\right]^{\frac{1}{2}}$, $k = 1 + |a_1| + |a_2|$.

15. Let $\phi_1, \ldots, \phi_n$ be n linearly independent solution of $L(y) = 0$ on an interval I. If $c_1, \ldots, c_n$ are any constants, prove that $\phi = c_1\phi_1 + \cdots + c_n\phi_n$ is a solution and every solution may be represented in this form.

16. Prove that there exist n linearly independent solutions of $L(y)=y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_n(x)y=0$ on I.

17. If ϕ_1 is a solution of $L(y)=y''+a_1(x)y'+a_2(x)y=0$ on an interval I and $\phi(x)\neq 0$ on I, a second solution ϕ_2 of $L(y)=y''+a_1(x)y'+a_2(x)y=0$ on I is given by $\phi_2(x)=\phi_1(x)\int_{x_0}^{x}\frac{1}{[\phi_1(s)]^2}\exp\left[-\int_{x_0}^{x}a_1(t)dt\right]ds$. Prove that the functions ϕ_1, ϕ_2 form a basis for the solutions of $L(y)=y''+a_1(x)y'+a_2(x)y=0$ on I.

18. Find the indicial polynomial for $L(y)=x^2y''+\frac{3}{2}xy'+xy=0$.

19. Suppose S is either a rectangle $|x-x_0|\le a$, $|y-y_0|\le b$, $(a,b>0)$, or a strip $|x-x_0|\le a$, $|y|<\infty$, $(a>0)$ and that f is a real valued function defined on S such that $\frac{\partial f}{\partial y}$ exists, is continuous on S and $\left|\frac{\partial f}{\partial y}(x,y)\right|\le K$, $((x,y)\,in\,S)$, for some $K>0$. Prove that f satisfies a Lipschitz condition on S with Lipschitz constants K.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. If ϕ_1, ϕ_2 are two solutions of $L(y) = 0$ on an interval I containing a point x_0. Prove that $W(\phi_1, \phi_1)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2)(x_0)$.

21. Prove that the n solutions $\phi_1, \ldots, \phi_n$ on an interval I of $L(y) = 0$ are linearly independent on any interval I.

22. If $\phi_1, \ldots, \phi_n$ are n solutions of $L(y) = 0$ on an interval I, prove that they are linearly independent if and only if $W(\phi_1, \ldots, \phi_n)(x) \neq 0$ for all x in I.

23. Show that $x^{\frac{1}{2}} J_{\frac{1}{2}}(x) = \dfrac{\sqrt{2}}{\Gamma\left(\dfrac{1}{2}\right)} \sin x$.

24. Let M, N be two real valued functions which have continuous first partial derivatives on some rectangle $R : |x - x_0| \leq a$, $|y - y_0| \leq b$. Prove that the equation $M(x, y) + N(x, y) y' = 0$ is exact in R if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ in R.
