MAY 2023 **P/ID 37506/PMA2F**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Define the degree of K over F.

2. Define algebraic number.

3. Define root of multiplicity m.

4. Define simple extension.

5. Define Galois group.

6. Define automorphism of the field K.

7. Define finite field.

8. Prove that any two finite fields having the same number of elements are isomorphic.

9. Define solvable group.

10. Define division algebra.

11. Define splitting field.

12. Let F be a finite field, then prove that F has p^m elements, where the prime number p is the characteristic of F.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. If a, b are in K are algebraic over F, then prove that $a+b$ and ab are also algebraic over F.

14. State and prove Remainder theorem.

15. Prove that $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$, for any $f(x),\ g(x) \in F[x]$.

16. Let $f(x) \in F[x]$ be of degree $n \geq 1$, then prove that there is an extension E of F of degree at most $n!$ in which $f(x)$ has n roots.

17. Let F be a finite field with q elements and suppose that $F \subset K$, where K is also a finite field. Prove that K has q^n elements when $n = [K : F]$.

18. Prove that for every prime number p and every positive integer m there exists a field having p^m elements.

19. Prove that G is solvable if and only if $G^{(k)} = (e)$ for some integer k.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

21. Prove that the polynomial $f(x) \in F[x]$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a same nontrivial common factor.

22. If K is a field and if $\sigma_1, \ldots, \sigma_n$ are distinct automorphisms of K, then prove that it is possible to find $a_1, \ldots, a_n$ not all 0, in K such that $a_1\sigma_1(u) + a_2\sigma_2(u) + \cdots + a_n\sigma_n(u) = 0$ for all $u \in K$.

23. State and prove Wedderburn's theorem.

24. State and prove Frobenius theorem.

———————