**MAY 2024** **P/ID 37506/PMA2F**

Time : Three hours Maximum : 80 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. When an element $a \in K$ is said to be algebraic over $F$.

2. Define algebraic of degree $n$.

3. Define splitting field.

4. Define simple extension.

5. Define Galois group.

6. Define the group of automorphisms of $K$ relative to $F$.

7. Define finite field.

8. If the finite field F has $p^m$ elements then prove that every $a \in F$ satisfies $a^{p^m} = a$.

9. Define solvable group.

10. When a division algebra $D$ is said to be algebraic over a field $F$.

11. Define fixed field of $G$.

12. Let $F$ be a finite field, then prove that $F$ has $p^m$ elements, where the prime number $p$ is the characteristic of $F$.

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. If $L$ is a finite extension of $F$ and if $K$ is a subfield of $L$ which contains $F$, then prove that $[K : F] / [L : F]$.

14. State and prove Remainder theorem.

15. If $K$ is a normal extension of $F$ then prove that $K$ is the splitting field of some polynomial over $F$.

16. Let $f(x) \in F[x]$ be of degree $n \geq 1$, prove that there is an extension $E$ of $F$ of degree at most $n!$ in which $f(x)$ has $n$ roots.

17. Prove that $G(K, F)$ is the subgroup of all automorphisms of $K$.

18. Prove that for every prime number $p$ and every positive integer $m$ there exists a field having $p^m$ elements.

19. Prove that the general polynomial of degree $n \geq 5$ is not solvable by radicals.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. Prove that the number $e$ is transcendental.

21. Prove that a polynomial of degree $n$ over a field can have at most $n$ roots in any extension field.

22. State and prove the fundamental theorem of Galois theory.

23. State and prove Wedderburn's theorem.

24. State and prove Frobenius theorem.

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