MAY 2025 **P/ID 37506/PMA2F**

Time : Three hours Maximum : 80 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Define algebraic extension of field F.

2. Find the dimension of $\mathbb{Q}(i)$ over $\mathbb{Q}$.

3. State remainder theorem.

4. Define the derivative of the polynomial $f(x) \in F[x]$ of degree n.

5. Prove that $(\alpha f(x))' = \alpha f'(x)$ for $f(x) \in F[x]$ and any $\alpha \in F$.

6. Define fixed field of group of automorphisms of K.

7. Define symmetric rational function.

8. Define primitive root.

9. State Euler's criterion.

10. Define solvable group.

11. Show that (12) and (12345) generate S_5.

12. When a division algebra is algebraic over a field F?

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. Prove that L is an algebraic extension of F if L is an algebraic extension of K and K is an algebraic extension of F.

14. Let $f(x) \in F[x]$ of $\deg n \geq 1$. Prove that there exists an extension E of F of degree at most $n!$ in which $f(x)$ has n roots.

15. If $f(x) \in F(x)$ is irreducible and the characteristics of F is $p \neq 0$, prove that $f(x)$ has a multiple root if it is of the form $f(x) = g(x^p)$.

16. Prove that the fixed field of G is a subfield of K.

17. If F is a finite field and $\alpha \neq 0, \beta \neq 0$ are two element F, prove that we can find elements a and b in F such that $1 + \alpha a^2 + \beta b^2 = 0$.

18. If G is a solvable group and $\overline{G}$ is a homomorphic image of G. prove that $\overline{G}$ is solvable.

19. Let C be the field of complex numbers and suppose that the division ring D is algebraic over C. Prove that $D = C$.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

21. Suppose F is of characteristics 0 and a,b are algebraic over F. Prove that there exists $c \in F(a,b)$ such that $F(a,b) = F(c)$.

22. State and prove fundamental theorem of Galois theory.

23. State and prove Wedderburn's theorem.

24. State and prove Frobenius theorem.

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