DECEMBER 2024 **P/ID 37507/PMA2G**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Define a Borel measurable function.

2. Define the Lebesgue outer measure.

3. Write Fatou's lemma.

4. Define the integral of a measurable function.

5. Write Jordan's test.

6. Define the Fourier series and Fourier coefficients of f relative to S.

7. State the Mean Value Theorem.

8. Write Dini's test.

9. If $f = u + iv$ is a complex valued function with a derivative at a point z in C, then prove that $J_f(z) = |f'(z)|^2$.

10. Write the first-order Taylor's formula.

11. If $x(r,\theta) = r\cos\theta$, $y(r,\theta) = r\sin\theta$, find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

12. Define stationary point and saddle point.

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. Prove that there exists a non-measurable set.

14. Evaluate $\lim \int_0^{\infty} \dfrac{dx}{\left(1+\dfrac{x}{n}\right)^{\alpha} x^{\frac{1}{n}}}$.

15. Let f be a bounded function defined on the finite interval [*a, b*]. Prove that f is Riemann integrable over [*a, b*] if and only if it is continuous a.e.

16. State and prove Riemann's Localization theorem.

17. Given $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$.

Prove that $D_{1,2}f(x,y)$ is not necessarily the same as $D_{2,1}f(x,y)$.

18. Let f be a real valued function with continuous second-order partial derivatives at a stationary point a in $\mathbb{R}^2$. Let $A = D_{1,1}f(a)$, $B = D_{1,2}f(a)$, $C = D_{2,2}f(a)$, and let $\Delta = \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = AC - B^2$.

Then prove that

(a) If $\Delta > 0$ and $A > 0$, f has a relative minimum at a

(b) If $\Delta > 0$ and $A < 0$, f has a relative maximum at a

(c) If $\Delta < 0$, f has a saddle point at a.

19. A quadric surface with center at the origin has the equation

$$Ax^2 + By^2 + cz^2 + 2Dyz + 2Ezx + 2Fxy = 1.$$

Find the lengths of its semi-axes.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Prove that the outer measure of an interval equals its length.

21. Let f be bounded and measurable on a finite interval $[a,b]$ and let $\in > 0$. Prove that there exists

(a) a step function h such that $\int_a^b |f - h| dx < \in$

(b) a continuous function g such that g vanishes outside a finite interval and $\int_a^b |f - g| dx < \in$.

22. Let $\{\varphi_0, \varphi_1, \varphi_2, ...\}$ be orthonormal on I and assume that $f \in L^2(I)$. Define two sequences of functions $\{s_n\}$ and $\{t_n\}$ on I as follows

$$s_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x), \quad t_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x)$$

where $c_k = (f, \varphi_k)$ for k = 0, 1, 2,... and $b_0, b_1, b_2, ...$ are arbitrary complex numbers. Prove that

$$\|f - s_n\| \leq \|f - t_n\|,$$

for each n. Further prove that equality holds if and only if $b_k = c_k$ for k = 0, 1, . . . , n.

23. If both the partial derivatives $D_{r,k}f$ and $D_{k,r}f$ exist in an n-ball $B(c, \delta)$ and if both are differentiable at c, then prove that $D_{r,k}f(c) = D_{k,r}f(c)$.

24. State and prove the inverse function theorem.

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