**MAY 2022** **P/ID 37507/PMA2G**

Time : Three hours Maximum : 80 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Define Lebesgue measurable set.
2. Define Borel function.
3. Define modulus of continuity.
4. State Lebesgue's monotone convergence theorem.
5. Define positive and negative parts of $f$.
6. Define Fourier series of $f$ relative of $S$.
7. State Dini's theorem.
8. State Riemann-Lebesgue Lemma.
9. Write the first order Taylor formula.
10. State the Chain Rule.
11. If $f = u + iv$ is a complex-valued function with a derivative at $a$ point $z$ in $C$, then prove that $J_f(z) = |f'(z)|^2$.
12. Define open mapping.

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. For any sequence of sets $\{E_i\}$, prove that $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$.

14. Prove that there exists a non-measurable set.

15. State and prove Lebesgue's Dominated Convergence Theorem.

16. Let $\{\phi_0, \phi_1, \phi_2, \cdots\}$ be orthonormal on $I$, and assume that $f \in L^2(I)$. Define two sequence of functions $\{s_n\}$ and $\{t_n\}$ on $I$ as follows :

$$s_n(x) = \sum_{k=0}^{n} c_k \phi_k(x), \quad t_n(x) = \sum_{k=0}^{n} b_k \phi_k(x),$$

where $c_k = (f, \phi_k)$, for $k = 0,1,2,\cdots$, and $b_0, b_1, b_2, \cdots$ are arbitrary numbers. For each $n$, prove that $\|f - s_n\| \leq \|f - t_n\|$.

17. State and prove Riemann's Localization Theorem.

18. Let $u$ and $v$ be two real-valued functions defined on a subset $S$ of the complex plane. Assume also that $u$ and $v$ are differentiable at an interior point $c$ of $S$ and that the partial derivatives satisfy the Cauchy-Riemann equations at $c$. Then prove that the function $f = u + iv$ has a derivative at $c$ and $f'(c) = D_1 u(c) + iD_1 v(c)$.

19. Assume that $f = (f_1, \cdots, f_n)$ has continuous partial derivatives $D_j f_i$ on an open set $S$ in $R^n$, and that the Jacobian determinant $J_f(a) \neq 0$, for some point $a$ in $S$. Prove that there is an $n$-ball $B(a)$ on which $f$ is one-to-one.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Prove that the class $\mathcal{M}$ is a $\sigma$-algebra.

21. If $f$ is Riemann integrable and bounded over the finite interval $[a,b]$, then prove that $f$ is integrable and $\mathbf{R}\int_a^b f dx = \int_a^b f\, dx$.

22. State and prove Jordan's Thoerem.

23. Assume that one of the partial derivatives $D_1\mathbf{f}, \cdots, D_n\mathbf{f}$ exists at $c$ and that the remaining $n-1$ partial derivatives exist in some $n$–ball $B(c)$ and are continuous at $\boldsymbol{c}$. Then prove that $\mathbf{f}$ is differentiable at $\mathbf{c}$.

24. State and prove the Inverse Function Theorem.

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