MAY 2023 **P/ID 37507/PMA2G**

Time : Three hours Maximum : 80 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer any TEN questions

1. Define σ–algebra.

2. Define Lebesgue measurable function.

3. Define Hausdorff outer measure.

4. Define positive and negative parts of f.

5. State Lebesgue's dominated convergence theorem.

6. Define orthonormal set.

7. State Dini's test.

8. State Riemann's Localization Theorem.

9. Define directional derivative of $\boldsymbol{f}$ at c in the direction u.

10. Define Jacobian Matrix.

11. Define stationary point.

12. Write the second-derivative test for extrema.

SECTION B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions

13. Prove that every interval is measurable.

14. Let c be any real number and let f and g be real-valued measurable function defined on the same measurable set E. Then prove that $f + c,\ cf,\ f + g,\ f - g$ and fg are also measurable.

15. Show that $\lim \int_0^\infty \frac{dx}{(1 + x/n)^n\ x^{1/n}} = 1.$

16. State and prove Riemann – Lebesgue Lemma.

17. State and prove Weierstrass Approximation Theorem.

18. State and prove Taylor's formula.

19. Assume that $f = (f_1, ..., f_n)$ has continuous partial derivatives $D_j f_i$ on an open set S in R^n, and that the jacobian determinant $J_f(a) \neq 0,$ for some point a in S. Prove that there is an n-ball $B(a)$ on which f is one-to-one.

SECTION C — ($3 \times 10 = 30$ marks)

Answer any THREE questions

20. For any set A, prove that $H_1^*(A) = m^*(A)$.

21. State and prove Fatou's Lemma.

22. State and prove Fejer's Theorem.

23. If both partial derivatives $D_r f$ and $D_k f$ exist in an n–ball $B(c;\delta)$ and if both are differentiable at c, then prove that $D_{r,k} f(c) = D_{k,r} f(c)$.

24. Let f be a real-valued function such that $f \in C'$ on an open set S in R^n. Let $g_1,.....g_m$ be m real valued functions such that $g = (g_1,....g_m) \in C'$ on S and assume that $m < n$. Let X_0 be that subset of S on which g vanishes, that is $X_0 = \{x : x \in S,\ g(x) = 0\}$, Assume that $x_0 \in X_0$ and assume that there exists an n-ball $B(x_0)$ such that $f(x) \le f(x_0)$ for all x in $X_0 \cap B(x_0)$ or such that $f(x) \ge f(x_0)$ for all x in $X_0 \cap B(x_0)$. Assume also

that the m– rowed determinant $\det[D_j g_i(x_0)] \neq 0$.

Then prove that there exist m real numbers $\lambda_1,.....,\lambda_m$ such that the following n equations are satisfied:

$$D_r f(x_0) + \sum_{k=1}^{m} \lambda_k D_r g_k(x_0) = 0 \ \ (r = 1,2,...n).$$