MAY 2024 **P/ID 37507/PMA2G**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Define Lebesgue measurable set.

2. Define Borel measurable function.

3. State Lebesgue's Monotone Convergence Theorem.

4. State Lebesgue's Dominated Convergence Theorem.

5. Write Dirichlet's kernel.

6. Write Jordan's test.

7. State the Weierstrass Approximation Theorem.

8. State the Mean Value Theorem.

9. Define saddle point.

10. If $f = u + iv$ is a complex valued function with a derivative at a point z in C, then prove that $J_f(z) = \left|f'(z)^2\right|$.

11. Define open mapping.

12. If $x(r,\theta) = r\cos\theta, y(r,\theta) = r\sin\ \theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Show that for any set A and any $\in > 0$ there is an open set O containing A and such that $m^* \le m^*(A) + \in$.

14. Prove that $\lim_{n\to\infty} \int_a^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2}\ dx = 0$ for $a > 0$ but not for $a = 0$.

15. State and prove Lebesgue's Monotone Convergence Theorem.

16. State and prove the Weierstrass approximation theorem.

17. Let S be an open connected subset of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}^n$ be differentiable at each point of S. If $f'(c) = 0$ for each c in S, then prove that f is constant on S.

18. For some integer $n \geq 1$, let f have a continuous n-th derivative in the open interval (a,b). Suppose also that for some interior point c in (a,b), we have

$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$, but $f^{(n)}(c) \neq 0$.

Then prove that for n even, f has a local minimum at c, if $f^{(n)}(c) > 0$, and a local maximum at c if $f^{(n)}(c) < 0$. If n is odd, then prove also that there is neither a local maximum nor a local minimum at c.

19. A quadric surface with center at the origin has the equation

$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 1$.

Find the lengths of its semi-axes.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Prove that the class $\mathcal{M}$ is a σ-algebra.

21. State and prove Fatou's lemma.

22. State and prove Jordan's theorem.

23. State and prove the Chain rule.

24. Let f be a real-valued function such that $f \in C'$ on an open set S in $\mathbb{R}^n$. Let $g_1,\ldots,g_m$ be m-real valued functions such that $g = (g_1,\ldots,g_m) \in C'$ on S and assume that $m < n$. Let X_0 be the subset of S on which g vanishes. Assume that $x_0 \in X_0$ and assume that there exists an n-ball $B(x_0)$ such that $f(x) \leq f(x_0)$ for all x in $X_0 \cap B(x_0)$ or such that $f(x) \geq f(x_0)$ for all x in $X_0 \cap B(x_0)$. Assume also that the m-rowed determinant $\det[D_j\ g_i(x_0)] \neq 0$. Prove that there exists m real numbers $\lambda_1,\ldots,\lambda_m$ such that the following n equations are satisfied:

$$D_r f(x_0) + \sum_{k=1}^{m} \lambda_k D_r gk(x_0) = 0, \quad (r = 1, 2,\ldots,n).$$