**MAY 2025** **P/ID 37507/PMA2G**

Time : Three hours Maximum : 80 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Define the Lebesgue outer measure.

2. Define a Borel measurable function.

3. Write Fatou's lemma.

4. State Lebesgue's Monotone convergence Theorem.

5. Write Dirichlet's kernel.

6. Define the Fourier series and Fourier coefficients of $f$ relative to $S$.

7. State the Weierstrass Approximation Theorem.

8. State the Mean Value Theorem.

9. If $f = u + iv$ is a complex valued function with a derivative at a point $z$ in $C$, then prove that $J_f(z) = |f'(z)|^2$.

10. Write the first-order Taylor's formula.

11. Define an open mapping.

12. Define stationary point and saddle point.

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. Prove that not every measurable set is a Borel set.

14. Prove that $\lim_{n\to\infty} \int_a^\infty \frac{n^2 x e^{-n^2 x^2}}{1+x^2} dx = 0$ for $a > 0$, but not for $a = 0$.

15. Let $f$ be a bounded function defined on the finite interval $[a, b]$. Prove that $f$ is Riemann integrable over $[a, b]$ if and only if it is continuous *a.e.*

16. State and prove Fejér's theorem.

17. *Let* $S$ be an open connected subset of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}^n$ be differentiable at each point $S$. If $f'(c) = 0$ for each $c$ in $S$, then prove that $f$ is constant on $S$.

18. Let $f$ be a real valued function with continuous second-order partial derivatives at a stationary point '$a$' in $\mathbb{R}^2$. Let

$A = D_{1,1} f(a),\ B = D_{1,2} f(a),\ C = D_{2,2} f(a)$, and let $\Delta = AC - B^2$.

Then prove that

(a) If $\Delta > 0$ and $A > 0$, $f$ has a relative minimum at $a$.

(b) If $\Delta > 0$ and $A < 0$, $f$ has a relative maximum at $a$

(c) If $\Delta < 0$, $f$ has a saddle point at $a$

19. State and prove the Second-derivative test for extrema.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Prove that the class $\mathcal{M}$ is a $\sigma$-algebra.

21. Let $f$ be bounded and measurable on a finite interval $[a, b]$ and let $\in > 0$. Prove that there exists.

    (a) a step function $h$ such that

    $$\int_a^b |f - h| dx < \in$$

    (b) a continuous function $g$ such that $g$ vanishes outside a finite interval and

    $$\int_a^b |f - g| dx < \in .$$

22. State and prove the Riesz-Fischer Theorem.

23. State and prove the Chain rule.

24. State and prove the inverse function theorem.

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