Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Prove that the basis given by interior of circles, on the plane $\mathbb{R}^2$ and the basis given by the interior of rectangles generate the same topology on the plane $\mathbb{R}^2$.

2. Using the basis of the topological spaces, X, Y construct the basis of the topology on the product space $X \times Y$.

3. Prove that any finite set in a Hansdroff space is closed.

4. Prove that Q – the set of rational numbers is not connected in $\mathbb{R}^2$ with usual topology.

5. Prove that the image of a connected set under a continuous map is connected.

6. State the Lebesgue number lemma.

7. State the first countability axiom.

8. Define the Cartesian product of the family of topological spaces.

9. Define a homotopy between two continuous maps $f, g : X \to Y$.

10. Define the product of paths in X.

11. State the properties satisfied by the elements in $\prod_1(X, x)$.

12. Define a star convex set in $\mathbb{R}^2$. Prove that it is simply connected.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let J be the topology generated by B, a basis for X. Prove that it is a topology on X.

14. Let Y be a subspace of X. Prove that a Set A in Y is closed if and only if $A = C \cap Y$, where C is closed in X. Using this prove that if Y is closed in X, and if A is closed in Y, then A is closed in X.

15. X, Y, Z be connected topological spaces. Prove that the Cartesian product $X \times Y \times Z$ is connected.

16. Let Y be a subspace of X. Prove that Y is compact if and only if every covering of Y by open sets in X contains a finite sub collection covering Y.

17. Show that a subspace of a completely regular space is completely regular. Also, product of completely regular spaces is completely regular.

18. Let $\{X_\alpha\}$ be an indexed family of spaces.

Let $A_\alpha \subset X_\alpha$. If $\prod X_\alpha$ has a box topology, show that $\prod \overline{A}_\alpha = \overline{\prod A_\alpha}$.

19. Prove that every metrizable space is normal.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. Let $f : X \to Y$. Prove that the following are equivalent.

(a) f is continuous

(b) Let $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$

(c) $B \subset Y$ – closed in Y, $f^{-1}(B)$ – closed in X.

(d) $\forall x \in X$, any neighborhood V of $f(x)$, $\exists a$ neighborhood U of x such that $f(U) \subset V$.

21. State the tube lemma. Prove that the product of two compact spaces is compact.

22. Prove that a regular space with a countable basis is normal.

23. State and prove the Tychonoff's theorem.

24. Define $\hat{\alpha}$, $h*$ where α is a path in X, $h: X \to Y$ is continuous such that $f(x_0) = y_0$. Prove the well definedness . Show $\hat{\alpha}$ is an isomorphism of groups. Show $h*$ is an isomorphism if h is a homeomorphism.

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