DECEMBER 2022 **P/ID 37517/PMA4T**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. What is a Curve?

2. Define Unit speed curve.

3. Define tangent space.

4. Define isometric.

5. Write the formula for the normal curvature.

6. Write the first and second fundamental forms of a surface σ.

7. Define Geodesic.

8. Write the geodesic patch.

9. Write the Gauss Equations.

10. Write the Codazzi-Mainardi Equations.

11. Define curvature.

12. Define are length.

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. Prove that any reparametrisation of a regular curve is regular.

14. Let $\sigma: U \to R^3$ be a patch of a surface S Containing a point P of S, and let (u,v) be coordinates in U. Prove that the tangent space to S at P is the vector space of R^3 spanned by the vectors σ_u and σ_v (the derivatives are evaluated at the point $(u_0, v_0) \in$ U such that $\sigma(u_0, v_0) = P$) .

15. Prove that $\|\sigma_u \times \sigma_v\| = (EG - F^2)^{1/2}$.

16. State and prove Meusnier's theorem.

17. Let $\sigma(u, v)$ be a surface path with first and second fundamental forms $Edu^2 + 2F\,du\,dv + G\,dv^2$ and $Ldu^2 + 2M\,du\,dv + N\,dv^2$ respectively. Then prove the following

(a) $K = \dfrac{LN - M^2}{EG - F^2}$

(b) $H = \dfrac{LG - 2MF + NE}{2(EG - F^2)}$

(c) The principal curvatures are $H \pm \sqrt{h^2 - K}$.

18. A curve γ on a surface S is a geodesic if and only if, for any part $\gamma(t)=\sigma(u(t),v(t))$ of γ contained in a surface patch σ of S, then prove the following :

$$\frac{d}{dt}(E\dot{u}+F\dot{v})=\frac{1}{2}\left(E_u\dot{u}^2+2F_u\dot{u}\dot{v}+G_u\dot{v}^2\right),$$

$$\frac{d}{dt}(F\dot{u}+E\dot{v})=\frac{1}{2}\left(E_v\dot{u}^2+2F_v\dot{u}\dot{v}+G_v\dot{v}^2\right),$$

where $Edu^2+2F\,du\,dv+G\,dv^2$ is the fundamental form of σ.

19. Prove that

(a) If $F=0$ then

$$K=\frac{1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)+\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right\}$$

(b) If $E=1$ and $F=0$, then $K=-\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}$.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Let γ be a unit-speed curve in R^3 with constant curvature and zero torsion. Then prove that γ is a circle.

21. State and prove Inverse function theorem.

22. Prove that if S is a compact surface, then there is a point P of S at which its Gaussian curvature K is > 0.

23. Prove that the unit speed curve γ is a geodesic if and only if $\frac{d}{dt}\mathcal{L}(\tau)=0$ where $\tau=0$.

24. State and prove Gauss's Remarkable theorem.

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