DECEMBER 2022 **P/ID 37518/PMA4U**

Time : Three hours Maximum : 80 marks

SECTION A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Define Banach space.

2. Define Schauder basis.

3. Define Graph of a function F.

4. State Banach-Steinhaus theorem.

5. Define conjugate space.

6. Define dual basis.

7. Define orthonormal set.

8. Write Parseval's formula.

9. Define normal operator.

10. Define eigen spectrum.

11. Define resolvent set.

12. Define Hilbert space.

SECTION B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. Let X be a nls. If $E_1 \subset X$ is open and $E_2 \subset X$, then prove that $E_1 + E_2$ is open.

14. Prove that two nlss X and Y over K are linearly homeomorphic if and only if there exists a linear map $F : X \to Y$ which is onto and satisfies

$$\sigma \| x \| \leq \| F(x) \| \leq \beta \| x \|$$

for all $x \in X$ and some $\alpha, \beta > 0$.

15. Let $A \in BL(X)$. If $k \in K$ and $\| A \| < |k|$, then prove that $k \notin s(A)$, $(A - kI)^{-1} = \sum_{n=0}^{\infty} A^n / k^{n+1}$ and

$$\| (A - kI)^{-1} \| \leq \frac{1}{|k| - \| A \|}.$$

16. Let X and Y be nlss and $F \in BL(X,Y)$. Then prove that $\| F' \| = \| F \|$. If we let $T(F) = F'$, then prove that T is a linear isometry of $BL(X, Y)$ into $BL(Y', X')$.

17. Let (x_n) and (y_n) be sequences in a nls X. If $x_n \xrightarrow{w} x$, $y_n \xrightarrow{w} y$ in X and $k_n \to k$ in K, then prove that $x_n + y_n \xrightarrow{w} x + y$ and $k_n x_n \xrightarrow{w} kx$.

18. Derive polarization identity :

19. If $A, B \in BL(H)$ and $k \in K$. Then prove the following :

(a) $(A+B)^* = A^* + B^*$

(b) $(kA)^* = kA^*$

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Let X be nls. Then prove that the closed unit ball in X is compact if and only if X is finite dimensional.

21. State and prove the uniform boundedness principle.

22. State and prove the Helly's selection principle.

23. State and prove Bessel's inequality.

24. Let $A \in BL(H)$. Then prove that

(a) A is unitary iff $\| A(x) \| = \| x \|$ for all $x \in H$ and A is onto

(b) A is normal iff $\| A(x) \| = \| A^*(x) \|$ for all $x \in H$.

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