DECEMBER 2023 **P/ID 37518/PMA4U**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Prove that addition and scalar multiplication are jointly continuous in a normal space.

2. If Y is a subspace of a normed space X and if $Y^o \neq \phi$, prove that Y = X

3. If $\{a_1, ..., a_m\}$ is a linearly independent set in a normed space X, show that there are $f_1, ...f_m$ in X' such that $f_j(a_i) = \delta_{ij}, 1 \leq i, j \leq m.$

4. Let X and Y be normed spaces and $F: X \to Y$ be linear. If go F is continuous for every $g \in Y$'s prove that F is continuous.

5. If Z is a closed subspace of a normed space X, prove that the quotient map Q from X to X/Z is continuous.

6. Let X be a normed space and $A \in BL(x)$. If A is bounded below and surjective, prove that A is invertible.

7. Give an example to show that $x_n' \xrightarrow{w^*} x'$ in X' does not imply that $x_n' \to x'$ in X'

8. If X is a reflective normed space and if $J' : X' \to X''$ is the canonical embedding of X' into its second dual X'', show that J' is surjective.

9. If X is an inner product space and $x \in X$, prove that $\langle x, y \rangle = 0$ for all $y \in X$ if and only if $x = 0$.

10. If X is an inner product space, prove that $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

11. If H is a Hilbert space and if $A \in BL(H)$, then prove that $\|A^*\| = \|A\|$

12. If $A \in BL(H)$ is normal and if x_1 and x_2 are eigen vectors of A corresponding to distinct eigen values, prove that $x_1 \perp x_2$.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is continuous if and only if for every Cauchy sequence (x_n) in X, the sequence $(F(x_n))$ is Cauchy in Y.

14. If Y is subspace of a normed space X and if $g \in Y$, show that there is some $f \in X'$ such that $f|_y = g$ and $\|f\| = \|g\|$

15. State and prove uniform boundedness principal.

16. If X is a Banach space, prove that the set of all invertible operators is open in BL(X) and the map $A \to A^{-1}$ is continuous on BL(X)

17. If the dual X' of a normed space X is separable, prove that so is X.

18. In an inner product space, prove that $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$ for all $x, y \in X$

19. If $A \in BL(H)$ is a Hilbert–Schmidt operator, prove that A is compact.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. Let M be a closed subspace of a normed space X. If the norm of a coset $X + M$ in the quotient space X/M is defined by $\|x+M\| = \inf\{\|x+m\| : m \in M\}$, prove that X/M is a normed space. Moreover, if X is a Banach space, prove that so is X/M.

21. If X is a non zero Banach space over C and $A \in BL(X)$, prove that

(a) $\sigma(A)$ is nonempty

(b) $r_o(A) = \lim_{n\to\infty} \| A^n \|^{1/n}$

22. If Y is a closed subspace of a normed space X, show that X is reflexive if and only if. Y and X/Y are reflexive.

23. State and prove Gram–Schmidt orthonormalization theorem.

24. Let $H = L^2([a,b])$ and $A \in BL(H)$. Show that A is a Hilbert–Schmidt operator if and only if A is a Fredholm integral operator with Kernel $k(.,.) \in L^2([a,b]\times[a,b])$

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