DECEMBER 2024 **P/ID 37518/PMA4U**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Define normed space.

2. Define hounded linear operator.

3. Define resolvent set.

4. Define bounded operator.

5. When will you say that a normed space is weakly convergent?

6. Define reflexive normed linear space.

7. Define orthonormal set.

8. Write Parseval's formula.

9. Define unitary operator.

10. Define normal operator.

11. Define Hilbert space.

12. Define Banach space.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let X be a normed space, Y be a closed subspace of X and $Y \neq X$. Let r be a real number such that $0 < r < 1$. Prove that there exists some $x_r \in X$ such that $\|X_r\| = 1$ and $r < dist(x_r, Y) \leq 1$.

14. Let X be a normed space and E be a subset of X. Then prove that E is bounded in X if and only if $f(E)$ is bounded in K for every $f \in X'$.

15. Let (x_n') be a sequence in a normed space X. If (a) (x_n') is bounded and (b) $(x_n'(x))$ is a Cauchy sequence in K for each x in a subset of X whose span is dense in X, then prove that (x_n') is weak* convergent in X'. The converse holds if X is a Banach space.

16. State and prove Pythagoras theorem.

17. Let H be a Hilbert space and $A \in BL(H)$. Then prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$; $\langle A(x), y \rangle = \langle x, B(y) \rangle$.

18. Let X be a Banach space in the norm $\|\ \|$. Then prove that a norm $\|\ \|'$ on the linear space X is equivalent to the norm $\|\ \|$ if and only if X is also a Banach space in the norm $\|\ \|'$ and the norm $\|\ \|'$ is comparable to the norm $\|\ \|$.

19. Let X be a normed space and $\{x_1', \ldots x_m'\}$ be a linearly independent subset of X', then prove that there are $x_1, \ldots, x_m$ in X such that $X_j'(x_i) = \delta_{i,j}$ for $j = 1, \ldots, m$.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. State and prove Hahn-Banach extension theorem.

21. Let X be a normed space and $A \in BL(X)$. Then prove that A is invertible if and only if A is bounded below and surjective.

If X is a Banach space. Then prove the following.

(a) A is invertible if and only if A is bijective.

(b) A is invertible if and only if A is bounded below and the range of A is dense in X.

22. Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. For a fixed $y \in l^q$, let $f_y(x) = \sum_{j=1}^{\infty} x(j)y(j)$, $x \in l^p$. Then prove that $f_y \in (l^p)'$ and $\|f_y\| = \|y\|_q$. The map $F : l^q \to (l^p)'$ defined by $F(y) = f_y$, $y \in l^q$, is a linear isometry from l^q into $(l^p)'$. If $1 \le p \le \infty$, then F is surjective. In fact, if $f \in (l^q)'$ and $y = (f(e_1), f(e_2), \ldots)$, then prove that $y \in l^q$ and $f = F(y)$.

23. Let $\{u_\alpha\}$ be an orthonormal set in an inner product space X and $x \in X$. Let $E_x = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$. Then prove that E_x is a countable set. If E_x is denumerable, then prove that $\langle x, u_n \rangle \to 0$ as $n \to \infty$. Further, if X is a Hilbert space, then $\sum_n \langle x, u_n \rangle$, converges in X to some y such that $x - y \perp u_\alpha$ for every α.

24. Let $A \in BL(H)$ be a Hilbert-Schmidt operator. Then prove that A is compact and A^* is a Hilbert-Schmidt operator.
