MAY 2022 **P/ID 37518/PMA4U**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Define normed linear space.
2. Define strictly convex nls.
3. Define graph of a function F.
4. Define bounded operator.
5. Define Conjugate space.
6. Define reflexive nls.
7. Define an inner product space.
8. Write Parseval's formula.
9. Define sesquilinear functional.
10. Define adjoint operator.
11. Define spectrum.
12. Define Banach space.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. Let X be a nls. If $f : X \to K$ is a linear functional, then prove that either $f \equiv 0$ or f maps open sets in X onto open sets in K.

14. If Y is a subspace of a nls X, then prove that Y and $\overline{Y}$ are nlss with the norm induced from X.

15. Let X be a nls, $U = \{x \in X : \| x \| < 1\}$ and $\delta > 1$. Let E be a subset of X such that for every $x \in U$ there exist $a \in E$ with $\| x - a \| < 1/\delta$. Then prove that for every $x \in U$ there exist a unique sequence (a_n) in E such that

$$\left\| x - \sum_{j=0}^{n} \left(a_j / \delta_j \right) \right\| < 1/\delta^{n+1}, \; n = 0, 1, 2, \cdots$$

16. Prove that the function which sends an invertible $A \in BL(X)$ to its inverse $A^{-1} \in BL(X)$ is continuous on the subset of invertible operators in $BL(X)$.

17. Let X and Y be nlss and $F \in BL(X, Y)$. Then prove that $F'' J_X = J_Y F$ and $\| F'' \| = \| F \|$, where $F'' = (F')'$ and J_x and J_Y are the canonical imbeddings of X and Y into X'' and Y'', respectively. Thus, F'' extends F linearly, preserving its norm.

18. Derive polarization identify.

19. If $A \in BL(H)$, then prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle A(x), y \rangle = \langle x, B(y) \rangle$.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. Let X and Y be nlss and $F : X \to Y$ a linear map. Prove that the following conditions are equivalent.

(a) F is bounded on same closed ball about 0 of positive radius.

(b) F is continuous at 0.

(c) F is continuous on X.

(d) F is uniformly continuous on X.

(e) $\|F(x)\| \leq \alpha \|x\|$ for every $x \in X$ and some $\alpha \in R$.

If in particular, $Y =$ **K**, then each of the above conditions is equivalent to

(f) the hyperspace $Z(F)$ is closed in X.

21. State and prove the closed graph theorem.

22. Let X be a reflexive nls. Then prove that

(a) Every closed subspace of X is reflexive.

(b) X is separable if and only if x' is separable.

23. State and prove the Gram-Schmidt orthonormalization theorem.

24. Let $A \in BL(H)$.

(a) If $K = C$, then prove that A is self-adjoint iff $\langle A(x), x \rangle \in R$ for all $x \in H$. Further, if $K = R$, then A is self-adjoint iff $\langle A(x), y \rangle = \langle A(y), x \rangle$ for all $x, y \in H$.

(b) A is unitary iff $\|A(x)\| = \|x\|$ for all $x \in H$ and A is onto.
