MAY 2023 **P/ID 37518/PMA4U**

Time : Three hours Maximum : 80 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Prove that the norm defined on a normed linear space is a continuous function.

2. Let X be a normed space. If E_1 is open in X and $E_2 \subseteq X$, prove that $E_1 + E_2$ is open in X.

3. If X is a normed space and if $0 \neq a \in X$, show that there is some $f \in X'$ such that $f(a) = \|a\|$ and $\|f\| = 1$.

4. Let X be a normed space and E be a subset of X. If E is bounded in X, prove that $f(E)$ is bounded in K for every $f \in X'$.

5. If a bijective map F from a metric space X to a metric space Y is closed, prove that its inverse F^{-1} is also a closed map.

6. Let X and Y be normed spaces. If $F \in BL(X,Y)$, prove that $\|F'\| = \|F\| = \|F''\|$.

7. If $x_n \xrightarrow{\omega} x$ in X and also $x_n \xrightarrow{\omega} y$ in X, prove that $y = x$.

8. If X is a reflexive normed space, prove that X' is reflexive.

9. Let x be an element in an inner product space X. Prove that $\langle x, y \rangle = 0$ for all $y \in X$ if and only if $x = 0$.

10. If $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, prove that $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

11. Let X be an inner product space and $A, B \in BL(X)$. If (A_n) and (B_n) are sequences in $BL(X)$ such that $A_n \to A$ and $B_n \to B$ in $BL(X)$, prove that $A_n B_n \to AB$.

12. Let H be a Hilbert space and $A \in BL(H)$. Prove that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$.

PART B — (5 × 6 = 30 marks)
Answer any FIVE questions.

13. Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on a linear space X. Prove that the norm $\|\cdot\|$ is stronger than the norm $\|\cdot\|'$ if and only if there is some $\alpha > 0$ such that $\|x\|^1 \leq \alpha \|x\|$ for all $x \in X$.

14. Let X and Y be normed space and $X \neq \{0\}$. Show that $BL(X,Y)$ is a Banach space in the operator norm if and only if Y is a Banach space.

15. If X and Y are normed spaces and if $F : X \to Y$ is linear, show that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$.

16. Let X be a Banach space, $A \in BL(X)$ and $\|A^p\| < 1$ for some positive integer p. Prove that (I-A) is invertible and find $\|(I - A)^{-1}\|$.

17. If $X = C([a,b])$ with the supnorm, show that $x_n \xrightarrow{\omega} x$ in X if and only if (x_n) is a bounded sequence in X and $x_n(t) \to x(t)$ for each $t \in [a,b]$.

18. If $X = C'([a,b])$ is the linear space of all scalar valued continuously differentiable functions on $[a,b]$, prove that X is an inner product space which is not a Hilbert space with respect to the function $\langle x, y \rangle_a = x(a)\overline{y}(a) + \int_a^b x'(t)\overline{y}'(t)\, dt$.

19. If $A \in BL(H)$ and if each A_n is compact, and $\|A_n - A\| \to 0$, prove that A is compact.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. If X is a normed space and Y is a finite dimensional subspace of X, prove that Y is complete.

21. If X and Y Banach spaces and if $F : X \to Y$ is a closed linear map, prove that F is continuous.

22. If X and Y Banach spaces and if $F \in BL(X,Y)$, show that $R(F) = Y$ if and only if F' is bounded below.

23. State and prove Riesz representation theorem.

24. If $A \in BL(H)$, show that

(a) $k \in \sigma(A)$ if that only if $\overline{k} \in \sigma(A^*)$

(b) $\sigma_e(A) \subset \sigma_a(A)$ and

$\sigma(A) = \sigma_0(A) \cup \{k : \overline{k} \in \sigma_e(A^*)\}$.

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