MAY 2024 **P/ID 37518/PMA4U**

Time : Three hours Maximum : 80 marks

PART A — ($10 \times 2 = 20$ marks)

Answer any TEN questions.

1. Prove that the norm is a continuous function.

2. Prove that K^n with the Euclidean norm $\| \|_2$ is strictly convex.

3. Let X be a normal space over K, and f be a non zero linear functional on X. If E is an open subset of X, prove that $f(E)$ is an open subset of K.

4. If E is a bounded subset of a normed space X, prove that $F(E)$ is bounded in K for every $f \in X!$.

5. Let X be a normed space and $P : X \to Y$ be a projection. If P is a closed mat, prove that $R(P)$ and $Z(P)$ are closed subspaces of X.

6. If $A \in BL(X)$ is invertible, prove that A is bounded below and surjective.

7. If X is a finite dimensional normed space, prove that its dual X' has the same dimension as X.

8. If $x_n \xrightarrow{w} x$ in X and also $x_n \xrightarrow{w} y$ in X, a normed space, prove that $x = y$.

9. In an inner product space X, if $x \in X$ prove that $< x, y >= 0$ for all $y \in X$ if and only if $x = 0$.

10. In an inner product space X, show that for all $x, y \in X$, $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

11. In an inner product space X, show that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$.

12. If $A \in BL(H)$ is normal, show that every spectral value of A is an approximate eigenvalue of A.

PART B — ($5 \times 6 = 30$ marks)

Answer any FIVE questions.

13. If $\|x\| = \left[\left(|x(1)^2| + |x(2)|^2\right)^{3/2} + |x(3)|^3\right]^{1/3}$, where $x = \left(x(1), x(2), x(3) \in K^3\right)$, prove that $\| \ \|$ is a norm on K^3.

14. Show that a normed space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X.

15. State and prove uniform boundedness principal.

16. If X and Y are Banach spaces and if $F : X \to Y$ is a linear map which is closed and surjective, prove that F is continuous and open.

17. Let $1 \le p \le \infty$ and $(1/p)+(1/q)=1$, then prove that the dual of C_{00} with the norm $\| \|_p$ is linearly isometric to l^q.

18. State and prove Bessel's inequality.

19. Show that every Hilbert-Schmidt operator defined on a Hilbert space is compact.

PART C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

20. If Y is a closed subspace of a normed space X, show that X is a Banach space if and only if Y and X/Y are Banach spaces in the induced norm and the quotient norm, respectively.

21. If X and Y are Banach spaces and if $F : X \to Y$ is a closed linear map, prove that F is continuous.

22. If X and Y are Banach spaces and if $F \in BL(X,Y)$, show that $R(F) = Y$ if and only if F' is bounded below.

23. State and prove the Riesz-representation theorem for a Hilbert space.

24. State and prove the spectral theorem for compact self-adjoint operators.
