MAY 2025 **P/ID 37518/PMA4U**

Time : Three hours Maximum : 80 marks

PART A — (10 × 2 = 20 marks)

Answer any TEN questions.

1. Define Euclidean norm.

2. Define strictly convex normed space.

3. Define continuous semi norm

4. When a bounded linear transformation is said to be invertible?

5. Define dual of a normed space X over K.

6. Define reflexive normed linear space.

7. Define Hilbert space.

8. Define orthonormal set.

9. Write Parseval's formula.

10. Define self-adjoint operator.

11. Define sesquilinear functional.

12. Define Banach space.

PART B — (5 × 6 = 30 marks)

Answer any FIVE questions.

13. Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range $R(F)$ of F is finite dimensional. Then prove that F is continuous if and only if the zero space $Z(F)$ of F is closed in X.

14. Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is closed and surjective. Then prove that F is continuous and open.

15. Let X, Y and Z be normed spaces. Let F_1 and F_2 be in $BL(X, Y)$ and $k \in K$. Then prove that $(F_1 + F_2)' = F_1' + F_2'$ and $(kF_1)' = k(F_1')$.

16. State and prove Pythagoras theorem.

17. Let A be a self adjoint operator on a finite dimensional Hilbert space H. Then prove that every root of the characteristic polynomial of A is real. In particular, A has an eigenvalue.

18. Let X be a normed space over K and f be a nonzero linear functional on X. If E is an open subset of X, then prove that $f(E)$ is an open subset of K.

19. Let E be a nonempty closed convex subset of a Hilbert space H. Then for each $x \in H$, there exists a unique best approximation from E to x. In particular, there is a unique element in E of minimal norm.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

20. State and prove Hahn-Banach extension theorem.

21. State and prove closed graph theorem.

22. Let X be a normed space and (x_n) be a sequence in X. Then prove that (x_n) is weak convergent in X if and only if

    (a) (x_n) is a bounded sequence in X and

    (b) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x' in some subset of X whose span is dense in x'. In that case, for every subsequence $(x_{n_k}\ of(x_n), x$ belongs to the closure of $co(\{x_{n1}, x_{n2}, \ldots\})$ and $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

23. Let X be an inner product space. Then prove the following.

(a) Let $E \subset X$ and $x \in \overline{E}$. Then there exists a best approximation from E to x if and only if $x \in E$.

(b) If $E \subset X$ is convex, then there exists at most one best approximation from E to any $x \in X$.

(c) Let F be a subspace of X and $x \in X$. Then $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$ and in that case $dist(x, F) = \langle x, x - y \rangle^{1/2}$.

24. Let H be a Hilbert space. Consider $A, B \in BL(H)$ and $k \in K$. Then prove that

(a) $(A + B)^* = A^* + B^*$.

(b) $(kA)^* = \overline{k}A^*$

(c) $(AB)^* = B^*A^*$

(d) $\left(A^*\right)^* = A$

(e) A is invertible if and only if A^* is invertible, and in that case $\left(A^*\right)^{-1} = \left(A^{-1}\right)^*$.
