

<b>Special Instructions/Useful Data</b>	
$\mathbb{N}$	Set of all natural numbers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers
$P^T$	Transpose of the matrix $P$
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n)^T \mid x_i \in \mathbb{R}, i = 1, 2, \dots, n\}$
$g'$	Derivative of a real valued function $g$
$g''$	Second derivative of a real valued function $g$
$P(A)$	Probability of an event $A$
i.i.d.	Independently and identically distributed
$N(\mu, \sigma^2)$	Normal distribution with mean $\mu$ and variance $\sigma^2$
$F_{m,n}$	$F$ distribution with $(m, n)$ degrees of freedom
$t_n$	Student's $t$ distribution with $n$ degrees of freedom
$\chi_n^2$	Central Chi-squared distribution with $n$ degrees of freedom
$\Phi(x)$	Cumulative distribution function of $N(0,1)$
$A^c$	Complement of a set $A$
$E(X)$	Expectation of a random variable $X$
$\text{Var}(X)$	Variance of a random variable $X$
$\text{Cov}(X, Y)$	Covariance between random variables $X$ and $Y$
$r!$	Factorial of an integer $r > 0, 0! = 1$
$\Phi(0.25) = 0.5987, \Phi(0.5) = 0.6915, \Phi(0.625) = 0.7341, \Phi(0.71) = 0.7612,$ $\Phi(1) = 0.8413, \Phi(1.125) = 0.8697, \Phi(1.5) = 0.9332, \Phi(1.64) = 0.95,$ $\Phi(2) = 0.9772$	

**SECTION – A**  
**MULTIPLE CHOICE QUESTIONS (MCQ)**

**Q. 1 – Q.10 carry one mark each.**

Q.1 The imaginary parts of the eigenvalues of the matrix

$$P = \begin{pmatrix} 3 & 2 & 5 \\ 2 & -3 & 6 \\ 0 & 0 & -3 \end{pmatrix}$$

are

- (A) 0, 0, 0                      (B) 2, -2, 0                      (C) 1, -1, 0                      (D) 3, -3, 0

Q.2 Let  $u, v \in \mathbb{R}^4$  be such that  $u = (1 \ 2 \ 3 \ 5)^T$  and  $v = (5 \ 3 \ 2 \ 1)^T$ . Then the equation  $uv^T x = v$  has

- (A) infinitely many solutions                      (B) no solution  
(C) exactly one solution                      (D) exactly two solutions

Q.3 Let  $u_n = \left(4 - \frac{1}{n}\right)^{\frac{(-1)^n}{n}}$ ,  $n \in \mathbb{N}$  and let  $l = \lim_{n \rightarrow \infty} u_n$ .

Which of the following statements is TRUE?

- (A)  $l = 0$  and  $\sum_{n=1}^{\infty} u_n$  is convergent  
(B)  $l = \frac{1}{4}$  and  $\sum_{n=1}^{\infty} u_n$  is divergent  
(C)  $l = \frac{1}{4}$  and  $\{u_n\}_{n \geq 1}$  is oscillatory  
(D)  $l = 1$  and  $\sum_{n=1}^{\infty} u_n$  is divergent

Q.4 Let  $\{a_n\}_{n \geq 1}$  be a sequence defined as follows:

$$a_1 = 1 \text{ and } a_{n+1} = \frac{7a_n + 11}{21}, n \in \mathbb{N}.$$

Which of the following statements is TRUE?

- (A)  $\{a_n\}_{n \geq 1}$  is an increasing sequence which diverges  
(B)  $\{a_n\}_{n \geq 1}$  is an increasing sequence with  $\lim_{n \rightarrow \infty} a_n = \frac{11}{14}$   
(C)  $\{a_n\}_{n \geq 1}$  is a decreasing sequence which diverges  
(D)  $\{a_n\}_{n \geq 1}$  is a decreasing sequence with  $\lim_{n \rightarrow \infty} a_n = \frac{11}{14}$

Q.5 Let  $X$  be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x^3, & \text{if } 0 < x \leq 1 \\ \frac{3}{x^5}, & \text{if } x > 1 \end{cases} .$$

Then  $P\left(\frac{1}{2} < X < 2\right)$  equals

- (A)  $\frac{15}{16}$                       (B)  $\frac{11}{16}$                       (C)  $\frac{7}{12}$                       (D)  $\frac{3}{8}$

Q.6 Let  $X$  be a random variable with the moment generating function

$$M_X(t) = \frac{1}{216}(5 + e^t)^3, \quad t \in \mathbb{R}.$$

Then  $P(X > 1)$  equals

- (A)  $\frac{2}{27}$                       (B)  $\frac{1}{27}$                       (C)  $\frac{1}{12}$                       (D)  $\frac{2}{9}$

Q.7 Let  $X$  be a discrete random variable with the probability mass function

$$p(x) = k(1 + |x|)^2, \quad x = -2, -1, 0, 1, 2,$$

where  $k$  is a real constant. Then  $P(X = 0)$  equals

- (A)  $\frac{1}{9}$                       (B)  $\frac{2}{27}$                       (C)  $\frac{1}{27}$                       (D)  $\frac{1}{81}$

Q.8 Let the random variable  $X$  have uniform distribution on the interval  $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ . Then  $P(\cos X > \sin X)$  is

- (A)  $\frac{2}{3}$                       (B)  $\frac{1}{2}$                       (C)  $\frac{1}{3}$                       (D)  $\frac{1}{4}$

Q.9 Let  $\{X_n\}_{n \geq 1}$  be a sequence of i.i.d. random variables having common probability density function

$$f(x) = \begin{cases} xe^{-x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} .$$

Let  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ ,  $n = 1, 2, \dots$ . Then  $\lim_{n \rightarrow \infty} P(\bar{X}_n = 2)$  equals

- (A) 0                      (B)  $\frac{1}{4}$                       (C)  $\frac{1}{2}$                       (D) 1

Q.10 Let  $X_1, X_2, X_3$  be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.$$

Which of the following estimators of  $\theta$  has the smallest variance for all  $\theta > 0$ ?

- (A)  $\frac{X_1+3X_2+X_3}{5}$                       (B)  $\frac{X_1+X_2+2X_3}{4}$   
 (C)  $\frac{X_1+X_2+X_3}{3}$                       (D)  $\frac{X_1+2X_2+3X_3}{6}$

**Q. 11 – Q. 30 carry two marks each.**

Q.11 Player  $P_1$  tosses 4 fair coins and player  $P_2$  tosses a fair die independently of  $P_1$ . The probability that the number of heads observed is more than the number on the upper face of the die, equals

- (A)  $\frac{7}{16}$                       (B)  $\frac{5}{32}$                       (C)  $\frac{17}{96}$                       (D)  $\frac{21}{64}$

Q.12 Let  $X_1$  and  $X_2$  be i.i.d. continuous random variables with the probability density function

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Using Chebyshev's inequality, the lower bound of  $P\left(|X_1 + X_2 - 1| \leq \frac{1}{2}\right)$  is

- (A)  $\frac{5}{6}$                       (B)  $\frac{4}{5}$                       (C)  $\frac{3}{5}$                       (D)  $\frac{1}{3}$

Q.13 Let  $X_1, X_2, X_3$  be i.i.d. discrete random variables with the probability mass function

$$p(k) = \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right), \quad k = 1, 2, 3, \dots$$

Let  $Y = X_1 + X_2 + X_3$ . Then  $P(Y \geq 5)$  equals

- (A)  $\frac{1}{9}$                       (B)  $\frac{8}{9}$                       (C)  $\frac{2}{27}$                       (D)  $\frac{25}{27}$

Q.14 Let  $X$  and  $Y$  be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} cx(1-x), & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases},$$

where  $c$  is a positive real constant. Then  $E(X)$  equals

- (A)  $\frac{1}{5}$                       (B)  $\frac{1}{4}$                       (C)  $\frac{2}{5}$                       (D)  $\frac{1}{3}$

Q.15 Let  $X$  and  $Y$  be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then  $P\left(X + Y > \frac{1}{2}\right)$  equals

- (A)  $\frac{23}{24}$                       (B)  $\frac{1}{12}$                       (C)  $\frac{11}{12}$                       (D)  $\frac{1}{24}$

Q.16 Let  $X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_n$  be i.i.d.  $N(0, 1)$  random variables. Then

$$W = \frac{n(\sum_{i=1}^m X_i)^2}{m(\sum_{j=1}^n Y_j^2)}$$

has

- (A)  $\chi_{m+n}^2$  distribution                      (B)  $t_n$  distribution  
 (C)  $F_{m,n}$  distribution                      (D)  $F_{1,n}$  distribution

Q.17 Let  $\{X_n\}_{n \geq 1}$  be a sequence of i.i.d. random variables with the probability mass function

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 4 \\ \frac{3}{4}, & \text{if } x = 8 \\ 0, & \text{otherwise} \end{cases}.$$

Let  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ ,  $n = 1, 2, \dots$ . If  $\lim_{n \rightarrow \infty} P(m \leq \bar{X}_n \leq M) = 1$ , then possible values of  $m$  and  $M$  are

- (A)  $m = 2.1, M = 3.1$                       (B)  $m = 3.2, M = 4.1$   
 (C)  $m = 4.2, M = 5.7$                       (D)  $m = 6.1, M = 7.1$

Q.18 Let  $x_1 = 1.1, x_2 = 0.5, x_3 = 1.4, x_4 = 1.2$  be the observed values of a random sample of size four from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} e^{\theta-x}, & \text{if } x \geq \theta \\ 0, & \text{otherwise} \end{cases}, \quad \theta \in (-\infty, \infty).$$

Then the maximum likelihood estimate of  $\theta^2$  is

- (A) 0.5                      (B) 0.25                      (C) 1.21                      (D) 1.44

- Q.19 Let  $x_1 = 2, x_2 = 1, x_3 = \sqrt{5}, x_4 = \sqrt{2}$  be the observed values of a random sample of size four from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{2\theta}, & \text{if } -\theta \leq x \leq \theta, \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.$$

Then the method of moments estimate of  $\theta$  is

- (A) 1                      (B) 2                      (C) 3                      (D) 4
- Q.20 Let  $X_1, X_2$  be a random sample from an  $N(0, \theta)$  distribution, where  $\theta > 0$ . Then the value of  $k$ , for which the interval  $(0, \frac{X_1^2 + X_2^2}{k})$  is a 95% confidence interval for  $\theta$ , equals
- (A)  $-\log_e(0.95)$       (B)  $-2 \log_e(0.95)$       (C)  $-\frac{1}{2} \log_e(0.95)$       (D) 2
- Q.21 Let  $X_1, X_2, X_3, X_4$  be a random sample from  $N(\theta_1, \sigma^2)$  distribution and  $Y_1, Y_2, Y_3, Y_4$  be a random sample from  $N(\theta_2, \sigma^2)$  distribution, where  $\theta_1, \theta_2 \in (-\infty, \infty)$  and  $\sigma > 0$ . Further suppose that the two random samples are independent. For testing the null hypothesis  $H_0: \theta_1 = \theta_2$  against the alternative hypothesis  $H_1: \theta_1 > \theta_2$ , suppose that a test  $\psi$  rejects  $H_0$  if and only if  $\sum_{i=1}^4 X_i > \sum_{j=1}^4 Y_j$ . The power of the test  $\psi$  at  $\theta_1 = 1 + \sqrt{2}, \theta_2 = 1$  and  $\sigma^2 = 4$  is

- (A) 0.5987              (B) 0.7341              (C) 0.7612              (D) 0.8413

- Q.22 Let  $X$  be a random variable having a probability density function  $f \in \{f_0, f_1\}$ , where

$$f_0(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_1(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

For testing the null hypothesis  $H_0: f \equiv f_0$  against  $H_1: f \equiv f_1$ , based on a single observation on  $X$ , the power of the most powerful test of size  $\alpha = 0.05$  equals

- (A) 0.425              (B) 0.525              (C) 0.625              (D) 0.725

- Q.23 If

$$\begin{aligned} & \int_{y=0}^1 \int_{x=y}^{2-\sqrt{1-(y-1)^2}} f(x, y) dx dy \\ &= \int_{x=0}^1 \int_{y=0}^{\alpha(x)} f(x, y) dy dx + \int_{x=1}^2 \int_{y=0}^{\beta(x)} f(x, y) dy dx, \end{aligned}$$

then  $\alpha(x)$  and  $\beta(x)$  are

- (A)  $\alpha(x) = x, \beta(x) = 1 + \sqrt{1 - (x - 2)^2}$       (B)  $\alpha(x) = x, \beta(x) = 1 - \sqrt{1 - (x - 2)^2}$   
 (C)  $\alpha(x) = 1 + \sqrt{1 - (x - 2)^2}, \beta(x) = x$       (D)  $\alpha(x) = 1 - \sqrt{1 - (x - 2)^2}, \beta(x) = x$

Q.24 Let  $f: [0,1] \rightarrow \mathbb{R}$  be a function defined as

$$f(t) = \begin{cases} t^3 \left(1 + \frac{1}{5} \cos(\log_e t^4)\right) & \text{if } t \in (0,1] \\ 0 & \text{if } t = 0 \end{cases}$$

Let  $F: [0,1] \rightarrow \mathbb{R}$  be defined as

$$F(x) = \int_0^x f(t) dt.$$

Then  $F''(0)$  equals

- (A) 0                      (B)  $\frac{3}{5}$                       (C)  $-\frac{5}{3}$                       (D)  $\frac{1}{5}$

Q.25 Consider the function

$$f(x,y) = x^3 - y^3 - 3x^2 + 3y^2 + 7, x, y \in \mathbb{R}.$$

Then the local minimum ( $m$ ) and the local maximum ( $M$ ) of  $f$  are given by

- (A)  $m = 3, M = 7$                       (B)  $m = 4, M = 11$   
 (C)  $m = 7, M = 11$                       (D)  $m = 3, M = 11$

Q.26 For  $c \in \mathbb{R}$ , let the sequence  $\{u_n\}_{n \geq 1}$  be defined by

$$u_n = \frac{\left(1 + \frac{c}{n}\right)^{n^2}}{\left(3 - \frac{1}{n}\right)^n}.$$

Then the values of  $c$  for which the series  $\sum_{n=1}^{\infty} u_n$  converges are

- (A)  $\log_e 6 < c < \log_e 9$                       (B)  $c < \log_e 3$   
 (C)  $\log_e 9 < c < \log_e 12$                       (D)  $\log_e 3 < c < \log_e 6$

Q.27 If for a suitable  $\alpha > 0$ ,

$$\lim_{x \rightarrow 0} \left( \frac{1}{e^{2x} - 1} - \frac{1}{\alpha x} \right)$$

exists and is equal to  $l$  ( $|l| < \infty$ ), then

- (A)  $\alpha = 2, l = 2$                       (B)  $\alpha = 2, l = -\frac{1}{2}$   
 (C)  $\alpha = \frac{1}{2}, l = -2$                       (D)  $\alpha = \frac{1}{2}, l = \frac{1}{2}$

Q.28 Let

$$P = \int_0^1 \frac{dx}{\sqrt{8 - x^2 - x^3}}.$$

Which of the following statements is TRUE?

- (A)  $\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) < P < \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{2}\right)$       (B)  $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{2}\right) < P < \sin^{-1}\left(\frac{1}{2}\right)$   
 (C)  $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) < P < \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$       (D)  $\sin^{-1}\left(\frac{1}{2}\right) < P < \frac{\sqrt{3}}{2}\sin^{-1}\left(\frac{1}{2}\right)$

Q.29 Let  $Q, A, B$  be matrices of order  $n \times n$  with real entries such that  $Q$  is orthogonal and  $A$  is invertible. Then the eigenvalues of  $Q^T A^{-1} B Q$  are always the same as those of

- (A)  $AB$                       (B)  $Q^T A^{-1} B$                       (C)  $A^{-1} B Q^T$                       (D)  $BA^{-1}$

Q.30 Let  $(x(t), y(t)), 1 \leq t \leq \pi$ , be the curve defined by

$$x(t) = \int_1^t \frac{\cos z}{z^2} dz \quad \text{and} \quad y(t) = \int_1^t \frac{\sin z}{z^2} dz.$$

Let  $L$  be the length of the arc of this curve from the origin to the point  $P$  on the curve at which the tangent is perpendicular to the  $x$ -axis. Then  $L$  equals

- (A)  $\sqrt{2}$                       (B)  $\frac{\pi}{\sqrt{2}}$                       (C)  $1 - \frac{2}{\pi}$                       (D)  $\frac{\pi}{2} + \sqrt{2}$

## SECTION - B

### MULTIPLE SELECT QUESTIONS (MSQ)

**Q. 31 – Q. 40 carry two marks each.**

Q.31 Let  $v \in \mathbb{R}^k$  with  $v^T v \neq 0$ . Let

$$P = I - 2 \frac{v v^T}{v^T v},$$

where  $I$  is the  $k \times k$  identity matrix. Then which of the following statements is (are) TRUE?

- (A)  $P^{-1} = I - P$                       (B)  $-1$  and  $1$  are eigenvalues of  $P$   
 (C)  $P^{-1} = P$                       (D)  $(I + P)v = v$

- Q.32 Let  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  be sequences of real numbers such that  $\{a_n\}_{n \geq 1}$  is increasing and  $\{b_n\}_{n \geq 1}$  is decreasing. Under which of the following conditions, the sequence  $\{a_n + b_n\}_{n \geq 1}$  is always convergent?
- (A)  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  are bounded sequences
- (B)  $\{a_n\}_{n \geq 1}$  is bounded above
- (C)  $\{a_n\}_{n \geq 1}$  is bounded above and  $\{b_n\}_{n \geq 1}$  is bounded below
- (D)  $a_n \rightarrow \infty$  and  $b_n \rightarrow -\infty$

- Q.33 Let  $f: [0,1] \rightarrow [0,1]$  be defined as follows:

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ x + \frac{2}{3}, & \text{if } x \in \mathbb{Q}^c \cap \left(0, \frac{1}{3}\right) \\ x - \frac{1}{3}, & \text{if } x \in \mathbb{Q}^c \cap \left(\frac{1}{3}, 1\right) \end{cases}$$

Which of the following statements is (are) TRUE?

- (A)  $f$  is one-one and onto
- (B)  $f$  is not one-one but onto
- (C)  $f$  is continuous on  $\mathbb{Q} \cap [0,1]$
- (D)  $f$  is discontinuous everywhere on  $[0,1]$
- Q.34 Let  $f(x)$  be a nonnegative differentiable function on  $[a, b] \subset \mathbb{R}$  such that  $f(a) = 0 = f(b)$  and  $|f'(x)| \leq 4$ . Let  $L_1$  and  $L_2$  be the straight lines given by the equations  $y = 4(x - a)$  and  $y = -4(x - b)$ , respectively. Then which of the following statements is (are) TRUE?
- (A) The curve  $y = f(x)$  will always lie below the lines  $L_1$  and  $L_2$
- (B) The curve  $y = f(x)$  will always lie above the lines  $L_1$  and  $L_2$
- (C)  $\left| \int_a^b f(x) dx \right| < (b - a)^2$
- (D) The point of intersection of the lines  $L_1$  and  $L_2$  lie on the curve  $y = f(x)$
- Q.35 Let  $E$  and  $F$  be two events with  $0 < P(E) < 1$ ,  $0 < P(F) < 1$  and  $P(E) + P(F) \geq 1$ . Which of the following statements is (are) TRUE?

- (A)  $P(E^c) \leq P(F)$
- (B)  $P(E \cup F) < P(E^c \cup F^c)$
- (C)  $P(E|F^c) \geq P(F^c|E)$
- (D)  $P(E^c|F) \leq P(F|E^c)$

Q.36 The cumulative distribution function of a random variable  $X$  is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{4}{9}, & \text{if } 0 \leq x < 1 \\ \frac{8}{9}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$$

Which of the following statements is (are) TRUE?

- (A) The random variable  $X$  takes positive probability only at two points  
 (B)  $P(1 \leq X \leq 2) = \frac{5}{9}$   
 (C)  $E(X) = \frac{2}{3}$   
 (D)  $P(0 < X < 1) = \frac{4}{9}$

Q.37 Let  $X_1, X_2$  be a random sample from a distribution with the probability mass function

$$f(x|\theta) = \begin{cases} 1 - \theta, & \text{if } x = 0 \\ \theta, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}, \quad 0 < \theta < 1.$$

Which of the following is (are) unbiased estimator(s) of  $\theta$ ?

- (A)  $\frac{X_1 + X_2}{2}$       (B)  $\frac{X_1^2 + X_2}{2}$       (C)  $\frac{X_1^2 + X_2^2}{2}$       (D)  $\frac{X_1 + X_2 - X_1^2}{2}$

Q.38 Let  $X_1, X_2, X_3$  be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.$$

If  $\delta(X_1, X_2, X_3)$  is an unbiased estimator of  $\theta$ , which of the following CANNOT be attained as a value of the variance of  $\delta$  at  $\theta = 1$ ?

- (A) 0.1      (B) 0.2      (C) 0.3      (D) 0.5

Q.39 Let  $X_1, X_2, \dots, X_n$  ( $n \geq 2$ ) be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} \frac{x}{\theta^2} e^{-x/\theta}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.$$

Let  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ . Which of the following statistics is (are) sufficient but NOT complete?

- (A)  $\bar{X}$       (B)  $\bar{X}^2 + 3$       (C)  $(X_1, \sum_{i=2}^n X_i)$       (D)  $(X_1, \bar{X})$

- Q.40 Let  $X_1, X_2, X_3, X_4$  be a random sample from an  $N(\theta, 1)$  distribution, where  $\theta \in (-\infty, \infty)$ . Suppose the null hypothesis  $H_0: \theta = 1$  is to be tested against the hypothesis  $H_1: \theta < 1$  at  $\alpha = 0.05$  level of significance. For what observed values of  $\sum_{i=1}^4 X_i$ , the uniformly most powerful test would reject  $H_0$ ?
- (A)  $-1$                       (B)  $0$                       (C)  $0.5$                       (D)  $0.8$

### SECTION – C

#### NUMERICAL ANSWER TYPE (NAT)

**Q. 41 – Q. 50 carry one mark each.**

- Q.41 Let the random variable  $X$  have uniform distribution on the interval  $(0, 1)$  and  $Y = -2 \log_e X$ . Then  $E(Y)$  equals \_\_\_\_\_
- Q.42 If  $Y = \log_{10} X$  has  $N(\mu, \sigma^2)$  distribution with moment generating function  $M_Y(t) = e^{5t+2t^2}$ ,  $t \in (-\infty, \infty)$ , then  $P(X < 1000)$  equals \_\_\_\_\_
- Q.43 Let  $X_1, X_2, X_3, X_4, X_5$  be independent random variables with  $X_1 \sim N(200, 8)$ ,  $X_2 \sim N(104, 8)$ ,  $X_3 \sim N(108, 15)$ ,  $X_4 \sim N(120, 15)$  and  $X_5 \sim N(210, 15)$ . Let  $U = \frac{X_1+X_2}{2}$  and  $V = \frac{X_3+X_4+X_5}{3}$ . Then  $P(U > V)$  equals \_\_\_\_\_
- Q.44 Let  $X$  and  $Y$  be discrete random variables with the joint probability mass function

$$p(x, y) = \frac{1}{25}(x^2 + y^2), \quad \text{if } x = 1, 2; y = 0, 1, 2.$$

Then  $P(Y = 1 | X = 1)$  equals \_\_\_\_\_

- Q.45 Let  $X$  and  $Y$  be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} 8xy, & 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then  $9\text{Cov}(X, Y)$  equals \_\_\_\_\_

- Q.46 Let  $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$  be i.i.d.  $N(\mu, \sigma^2)$  random variables. Let  $\bar{X} = \frac{1}{3} \sum_{i=1}^3 X_i$  and  $\bar{Y} = \frac{1}{4} \sum_{j=1}^4 Y_j$ . If  $k \sqrt{\frac{15}{7}} \frac{(\bar{X} - \bar{Y})}{\sqrt{\{\sum_{i=1}^3 (X_i - \bar{X})^2 + \sum_{j=1}^4 (Y_j - \bar{Y})^2\}}}$  has  $t_\nu$  distribution, then  $(\nu - k)$  equals \_\_\_\_\_

Q.47 Let  $f: \left[0, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$  be defined as

$$f(x) = \alpha x + \beta \sin x,$$

where  $\alpha, \beta \in \mathbb{R}$ . Let  $f$  have a local minimum at  $x = \frac{\pi}{4}$  with  $f\left(\frac{\pi}{4}\right) = \frac{\pi-4}{4\sqrt{2}}$ .

Then  $8\sqrt{2}\alpha + 4\beta$  equals \_\_\_\_\_

Q.48 The area bounded between two parabolas  $y = x^2 + 4$  and  $y = -x^2 + 6$  is \_\_\_\_\_

Q.49 For  $j = 1, 2, \dots, 5$ , let  $P_j$  be the matrix of order  $5 \times 5$  obtained by replacing the  $j^{\text{th}}$  column of the identity matrix of order  $5 \times 5$  with the column vector  $v = (5 \ 4 \ 3 \ 2 \ 1)^T$ . Then the determinant of the matrix product  $P_1 P_2 P_3 P_4 P_5$  is \_\_\_\_\_

Q.50 Let

$$u_n = \frac{18n + 3}{(3n - 1)^2(3n + 2)^2}, \quad n \in \mathbb{N}.$$

Then  $\sum_{n=1}^{\infty} u_n$  equals \_\_\_\_\_

**Q. 51 – Q. 60 carry two marks each.**

Q.51 Let a unit vector  $v = (v_1 \ v_2 \ v_3)^T$  be such that  $Av = 0$  where

$$A = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}.$$

Then the value of  $\sqrt{6}(|v_1| + |v_2| + |v_3|)$  equals \_\_\_\_\_

Q.52 Let

$$F(x) = \int_0^x e^t(t^2 - 3t - 5)dt, \quad x > 0.$$

Then the number of roots of  $F(x) = 0$  in the interval  $(0, 4)$  is \_\_\_\_\_

- Q.53 A tangent is drawn on the curve  $y = \frac{1}{3}\sqrt{x^3}$ , ( $x > 0$ ) at the point  $P\left(1, \frac{1}{3}\right)$  which meets the x-axis at  $Q$ . Then the length of the closed curve  $OQPO$ , where  $O$  is the origin, is \_\_\_\_\_

- Q.54 The volume of the region

$$R = \{(x, y, z) \in \mathbb{R}^3 : x + y + z \leq 3, y^2 \leq 4x, 0 \leq x \leq 1, y \geq 0, z \geq 0\}$$

is \_\_\_\_\_

- Q.55 Let  $X$  be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{x}{8}, & \text{if } 0 < x < 2 \\ \frac{k}{8}, & \text{if } 2 \leq x \leq 4 \\ \frac{6-x}{8}, & \text{if } 4 < x < 6 \\ 0, & \text{otherwise.} \end{cases}$$

where  $k$  is a real constant. Then  $P(1 < X < 5)$  equals \_\_\_\_\_

- Q.56 Let  $X_1, X_2, X_3$  be independent random variables with the common probability density function

$$f(x) = \begin{cases} 2e^{-2x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Let  $Y = \min\{X_1, X_2, X_3\}$ ,  $E(Y) = \mu_y$  and  $\text{Var}(Y) = \sigma_y^2$ . Then  $P(Y > \mu_y + \sigma_y)$  equals \_\_\_\_\_

- Q.57 Let  $X$  and  $Y$  be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{2}e^{-x}, & \text{if } |y| \leq x, x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Then  $E(X | Y = -1)$  equals \_\_\_\_\_

- Q.58 Let  $X$  and  $Y$  be discrete random variables with  $P(Y \in \{0, 1\}) = 1$ ,

$$\begin{aligned} P(X = 0) &= \frac{3}{4}, & P(X = 1) &= \frac{1}{4}, \\ P(Y = 1 | X = 1) &= \frac{3}{4}, & P(Y = 0 | X = 0) &= \frac{7}{8}. \end{aligned}$$

Then  $3P(Y = 1) - P(Y = 0)$  equals \_\_\_\_\_

Q.59 Let  $X_1, X_2, \dots, X_{100}$  be i.i.d. random variables with  $E(X_1) = 0$ ,  $E(X_1^2) = \sigma^2$ , where  $\sigma > 0$ . Let  $S = \sum_{i=1}^{100} X_i$ . If an approximate value of  $P(S \leq 30)$  is 0.9332, then  $\sigma^2$  equals \_\_\_\_\_

Q.60 Let  $X$  be a random variable with the probability density function

$$f(x|r, \lambda) = \frac{\lambda^r}{(r-1)!} x^{r-1} e^{-\lambda x}, \quad x > 0, \lambda > 0, r > 0.$$

If  $E(X) = 2$  and  $\text{Var}(X) = 2$ , then  $P(X < 1)$  equals \_\_\_\_\_

**END OF THE QUESTION PAPER**

# JAM 2017 ANSWER KEY

## Model Answer Key for MS Paper

Paper: MATHEMATICAL STATISTICS	Code: MS
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SECTION – A (MCQ)				SECTION – B (MSQ)		SECTION – C (NAT Type)			
Q. No.	KEY	Q. No.	KEY	Q. No.	KEYS	Q. No.	KEY RANGE	Q. No.	KEY RANGE
01	A	16	D	31	B, C	41	(2.0, 2.0)	56	(0.13, 0.14)
02	B	17	D	32	A, C	42	(0.15, 0.16)	57	(2.0, 2.0)
03	D	18	B	33	A, D	43	(0.97, 0.98)	58	(0.12, 0.13)
04	D	19	C	34	A, C	44	(0.25, 0.25)	59	(4.0, 4.0)
05	A	20	B	35	A, C, D	45	(0.15, 0.17)	60	(0.25, 0.27)
06	A	21	D	36	B, C	46	(3.0, 3.0)		
07	C	22	B	37	A, B, C	47	(4.0, 4.0)		
08	D	23	B	38	A, B, C	48	(2.6, 2.7)		
09	A	24	A	39	C, D	49	(120.0, 120.0)		
10	C	25	D	40	A, B, C	50	(0.25, 0.25)		
11	C	26	B			51	(4.0, 4.0)		
12	C	27	B			52	(0, 0)		
13	B	28	A			53	(2.1, 2.2)		
14	C	29	D			54	(2.1, 2.3)		
15	A	30	C			55	(0.87, 0.88)		